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Differential Subspaces Associated with Pairs of Ordinary Differential Expressions

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1. INTRODUCTION

In this paper we begin a study of boundary-value problems associated with a pair of ordinary differential expressions L and M of orders n and $\nu = 2\mu$, respectively, acting on vector-valued functions $f\colon \iota \to \mathbb{C}^m$, where $\iota = (a, b)$ is an arbitrary open real interval, and $\mathbb{C}$ is the complex number field. We assume M is positive on $C_0^\infty(\iota)$, the functions of class $C^\infty(\iota)$ which have compact support, i.e.,

$$(Mf, f)_2 = \int_\iota f^*(Mf) \geqslant 0, \qquad f \in C_0^\infty(\iota).$$

Then we can introduce an inner product $(\ ,\ )$ on $C_0^\infty(\iota)$ via $(f, g) = (Mf, g)_2$, and $\|f\| = (f, f)^{1/2}$ will be a norm there. The basic Hilbert space in which we study L and M is one generated by a positive self-adjoint extension H of M_0 (which is M on $C_0^\infty(\iota)$) in $L^2(\iota)$. It was shown by Krein [19] that each such extension satisfies $M_N \leqslant H \leqslant M_F$, where M_N and M_F are two special positive extensions of M_0, namely, the von Neumann and Friedrichs extension, respectively. For a recent treatment of this result for multivalued operators see [9]. We assume that such an extension satisfies a local inequality

$$(Hf, f)_2 \geqslant (c(J))^2 (f, f)_{2,J},$$

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for all $f \in \mathfrak{D}(H)$, the domain of H, where $c(J) > 0$ and J is any compact subinterval of ι. Then the completion of $\mathfrak{D}(H)$ with the inner product $(f, g) = (Hf, g)_2$ is a Hilbert space $\mathfrak{H} = \mathfrak{H}_H$, and we assume that the identity map on $\mathfrak{D}(H)$ extends to an injection of $\mathfrak{H}$ into $L^2_{\text{loc}}(\iota)$, the set of all f which are in $L^2(J)$ for each compact $J \subset \iota$. Now $C_0^\infty(\iota) \subset \mathfrak{H}_H$ and the inner product on $C_0^\infty(\iota)$ is a Dirichlet integral, whereas the inner product on $\mathfrak{D}(H)$ will in general involve boundary terms as well. The closure $(C_0^\infty(\iota))^c$ of $C_0^\infty(\iota)$ in $\mathfrak{H}$ is $\mathfrak{H}_M = \mathfrak{H}_{M_F}$, where M_F is the Friedrichs extension of M_0, and we have $\mathfrak{H}_H = \mathfrak{H}_M \oplus \mathfrak{N}_H$, an orthogonal sum, where

$$\mathfrak{N}_H = \{f \in C^\nu(\iota) \cap \mathfrak{H}_H \mid Mf = 0\}.$$

The injection $\mathfrak{H} \subset L^2_{\text{loc}}(\iota)$ implies the existence of an injection $G: L_0^2(\iota) \to \mathfrak{H}$, where $L_0^2(\iota)$ is the set of all $f \in L^2(\iota)$ which have compact support. This G satisfies

$$(f, G\alpha) = (f, \alpha)_2, \qquad f \in \mathfrak{H},\ \alpha \in L_0^2(\iota),$$
$$GM\varphi = \varphi, \qquad \varphi \in C_0^\infty(\iota), \qquad MG\alpha = \alpha, \qquad \alpha \in L_0^2(\iota),$$
$$(GL_0^2(\iota))^c = \mathfrak{H}.$$

If the self-adjoint operator H satisfies a global inequality

$$(Hf, f)_2 \geqslant c^2(f, f)_2, \qquad f \in \mathfrak{D}(H),$$

for some constant $c > 0$ then the identity map on $\mathfrak{D}(H)$ extends to an injection of $\mathfrak{H}$ into $L^2(\iota)$, and G has an extension (call it G) to all of $L^2(\iota)$ into $\mathfrak{H}$. It satisfies the above listed properties with $L_0^2(\iota)$ replaced by $L^2(\iota)$. In this situation $G = H^{-1}$, which is a bounded operator on $L^2(\iota)$, and $\mathfrak{H} = \mathfrak{D}(H^{1/2})$, the domain of the positive square root $H^{1/2}$ of H. An instance of this situation occurs in the regular case where ι is a finite interval and the coefficients of M are smooth on the closure $\bar{\iota}$ of ι. The local inequality then implies the global one. In a second paper we shall study in detail the regular case. It turns out that in this case $\mathfrak{H}$ is the set of all $f \in C^{\mu-1}(\bar{\iota})$, such that $f^{(\mu-1)}$ is absolutely continuous on $\bar{\iota}$, $f^{(\mu)} \in L^2(\iota)$, and where f satisfies a set of homogeneous boundary conditions in $f, f', \ldots, f^{(\mu-1)}$, the so-called essential boundary conditions for H. The inner product in this $\mathfrak{H}$ can be explicitly given; it involves a Dirichlet integral as well as boundary terms. This situation was extensively studied by Krein [20].

Associated with a given $\mathfrak{H} = \mathfrak{H}_H$, and the pairs L, M and L^+, M, where L^+ is the formal adjoint of L, are the maximal smooth linear manifolds T, T^+ in $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$,

$$T = \{\{f, g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), g \in C^\nu(\iota), Lf = Mg\},$$
$$T^+ = \{\{f, g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), g \in C^\nu(\iota), L^+f = Mg\},$$

where $r = \max(n, \nu)$. Then minimal linear manifolds are defined by

$$S = \{\{\varphi, GL\varphi\} \mid \varphi \in C_0^\infty(\iota)\},$$
$$S^+ = \{\{\varphi, GL^+\varphi\} \mid \varphi \in C_0^\infty(\iota)\}.$$

Now $S \subset T$, $S^+ \subset T^+$, and S, S^+ are (the graphs of) operators, whereas T, T^+ are not (the graphs of) operators in general, for

$$T(0) = \{g \in \mathfrak{H} \mid \{0, g\} \in T\} = T^+(0) = \mathfrak{N}_H .$$

The minimal and maximal subspaces (closed linear manifolds in $\mathfrak{H}^2$) are defined by $T_0 = S^c$, $T_0^+ = (S^+)^c$ and $T_1 = T^c$, $T_1^+ = (T^+)^c$, respectively. These can be considered as multivalued operators. We have $(T_0)^* = T_1^+$, $(T_0^+)^* = T_1$. Now $T_1 \ominus T_0$, the orthogonal complement of T_0 in T_1, and $T_1^+ \ominus T_0^+$, are contained in $C^r(\iota) \times C^r(\iota)$.

The central objects of study are the intermediate adjoint pairs of subspaces A, A^* satisfying

$$T_0 \subset A \subset T_1, \qquad T_0^+ \subset A^* \subset T_1^+,$$

with $\dim(A/T_0) = d$. Since $\dim(T_1 \ominus T_0) \leqslant 2rm$ we can characterize these A, A^* via a finite number of generalized boundary conditions. However, A and A^* in general contain nonsmooth elements, so that these boundary conditions cannot be written in the usual way. Smooth versions $A \cap T$, $A^* \cap T^+$, of A and A^* are such that $(A \cap T)^c = A$, $(A^* \cap T^+)^c = A^*$, and, in many situations, $A \cap T$ and $A^* \cap T^+$ may be characterized by boundary conditions of the usual sort. This will be shown in detail for the regular case in our next paper.

We also consider more general subspace extensions A of T_0, namely those satisfying

$$T_0 \subset A \subset \mathfrak{K}^2, \qquad T_0^+ \subset A^* \subset \mathfrak{K}^2,$$

where $\mathfrak{K}$ is a (perhaps larger) Hilbert space containing $\mathfrak{H}$. Assuming the resolvent set $\rho(A)$ is not empty, we define the generalized resolvent R_A of T_0 corresponding to A as the operator-valued function given by

$$R_A(l)h = P(A - lI)^{-1}h, \qquad h \in \mathfrak{H}, \, l \in \rho(A).$$

Here P is the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$; if $\mathfrak{K} = \mathfrak{H}$, R_A is the resolvent of A. We show that $R_A(l)G: L_0^2(\iota) \to \mathfrak{H}$ is an integral operator with a smooth kernel for all $l \in \rho(A) \cap (\mathbb{C} \backslash \mathbb{C}_e)$, where the exceptional set $\mathbb{C}_e$ is the set of all $l \in \mathbb{C}$, where the leading coefficient of $L - lM$ is not invertible for some $x \in \iota$. In case M is multiplication by the $m \times m$ unit matrix I_m, this implies $R_A(l)$ is an integral operator of Carleman type. The same methods show that G is an integral operator with a smooth kernel.

The results for the important case when S is symmetric are summarized; S is symmetric if and only if $L = L^+$.

A summary of some of the results presented here was given at the Czechoslovak Conference on Differential Equations and Their Applications, Equadiff 4, at Prague in August 1977 [8]. An outline of the detailed results in the regular case was presented at Equadiff 78, International Conference on Ordinary Differential Equations and Functional Equations, held in Florence, Italy in May 1978 [10].

We emphasize that we are dealing with systems where no restriction is placed on the relative orders of L and M; all cases $n > \nu$, $n = \nu$, $n < \nu$, are allowed. Thus we are considering left definite as well as right definite problems, for if a subspace S has M positive (the so-called right definite case), then S^{-1} will correspond to the left definite case. The consideration of problems of the left definite type goes back at least to Weyl [29, pp. 239–247] who considered second-order problems. There is a large literature devoted to problems for two differential expressions L and M. Kamke [17] and Brauer [2] considered problems in the regular case, where the Hilbert space is generated by a self-adjoint extension H of M_0 which differs from the Friedrichs extension M_F. In [3] Brauer treated a singular operator case with the global inequality and $\mathfrak{H} = \mathfrak{H}_{M_F}$, and Browder [5, 6] considered an $\mathfrak{H}$, which is a completion of C_0^∞, in connection with his eigenfunction expansion results for nonsymmetric partial differential operators. Both Pleijel [22, 23] and Bennewitz [1] deal with subspaces, but our basic Hilbert space $\mathfrak{H}$, whose inner product in general involves boundary as well as integral terms, differs from theirs. Other approaches have been used to study pairs L and M. For example, Brauer in [4] and Niessen in [21] considered first-order systems. Problems of the left definite type have also been considered by Shotwell [25], and by Schneider and Niessen [24]. Hilbert spaces, and more general spaces, generated by special differential operators M have been studied by Triebel [27].

We recall some basic notions concerning linear manifolds and subspaces (closed linear manifolds) in $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$, where $\mathfrak{H}$ is a Hilbert space over the complex numbers $\mathbb{C}$. If S, T are linear manifolds in $\mathfrak{H}^2$, we define

$$\begin{aligned}
\mathfrak{D}(S) &= \{f \in \mathfrak{H} \mid \{f, g\} \in S, \text{ some } g \in \mathfrak{H}\} = \text{domain of } S,\\
\mathfrak{R}(S) &= \{g \in \mathfrak{H} \mid \{f, g\} \in S, \text{ some } f \in \mathfrak{H}\} = \text{range of } S,\\
S(f) &= \{g \in \mathfrak{H} \mid \{f, g\} \in S\}, \qquad f \in \mathfrak{D}(S),\\
S^{-1} &= \{\{g, f\} \in \mathfrak{H}^2 \mid \{f, g\} \in S\},\\
\nu(S) &= \{f \in \mathfrak{H} \mid \{f, 0\} \in S\} = S^{-1}(0) = \text{null space of } S,\\
\alpha S &= \{\{f, \alpha g\} \mid \{f, g\} \in S\}, \qquad \alpha \in \mathbb{C},\\
S + T &= \{\{f, g + k\} \mid \{f, g\} \in S, \{f, k\} \in T\},\\
I &= \{\{f, f\} \in \mathfrak{H}^2\} = \text{identity operator on } \mathfrak{H},\\
TS &= \{\{f, g\} \in \mathfrak{H}^2 \mid \{f, h\} \in S, \{h, g\} \in T \text{ for some } h \in \mathfrak{H}\}.
\end{aligned}$$

Operators (single-valued functions) in $\mathfrak{H}$ will be identified with their graphs in $\mathfrak{H}^2$; they are the linear manifolds S such that $S(0) = \{0\}$. If S is an operator in $\mathfrak{H}$ we write Sf for $S(f)$. The algebraic sum $S \dotplus T$ is given by

$$S \dotplus T = \{\{f + h, g + k\} \mid \{f, g\} \in S, \{h, k\} \in T\}.$$

For any linear manifold M in a Hilbert space $\mathfrak{K}$ we denote by M^c its closure, and $M^\perp = \mathfrak{K} \ominus M$, the orthogonal complement of M in $\mathfrak{K}$. We have $\mathfrak{K} = M \oplus M^\perp$, an orthogonal algebraic sum, if M is a subspace in $\mathfrak{K}$. In $\mathfrak{H}^2$ we have the self-pairing

$$\langle\{f, g\}, \{h, k\}\rangle = (g, h) - (f, k), \qquad \{f, g\}, \{h, k\} \in \mathfrak{H}^2.$$

If S is a linear manifold in $\mathfrak{H}^2$, its adjoint S^* is defined by

$$S^* = \{\{h, k\} \in \mathfrak{H}^2 \mid \langle\{f, g\}, \{h, k\}\rangle = 0, \text{ all } \{f, g\} \in S\}.$$

It is a subspace in $\mathfrak{H}^2$, and in fact $S^* = (JS)^\perp$, where $J: \mathfrak{H}^2 \to \mathfrak{H}^2$ is the unitary map given by $J\{f, g\} = \{g, -f\}$. A linear manifold S is symmetric if $S \subset S^*$, and it is self-adjoint if $S = S^*$. Further details about subspaces in $\mathfrak{H}^2$ can be found in [7], or [11].

The real number field is denoted by $\mathbb{R}$, the $j \times k$ zero matrix by 0_j^k.

2. Preliminary Results on Systems of Differential Expressions

In this section we collect some basic results we shall require in the succeeding sections. First we present brief proofs of fairly general regularity results for systems of equations. The basic ideas for these go back at least to Halperin [16], and later Kemp [18] showed how to obtain such results even though the leading coefficient of the differential expression has zeros on the interval under consideration. Next we define the maximal and minimal operators in an L^2-space associated with a formal differential expression, and show what their domains are in the regular case of a finite interval. Some formal material on even-order symmetric operators is presented, since these will be the basic operators used in defining our Hilbert spaces.

We settle some notational matters. Consider an open interval $\iota = (a, b)$ on the real axis $\mathbb{R}$, where $a = -\infty$, $b = +\infty$, or both are allowed. For a fixed positive integer m let $C^0(\iota) = C(\iota)$ denote the set of all continuous $m \times 1$ matrix-valued functions f on ι whose components have values in $\mathbb{C}$, the complex numbers, i.e., $f: \iota \to \mathbb{C}^m$. For $k \in \mathbb{N}$ (the positive integers) let $C^k(\iota)$ denote those $f \in C(\iota)$ such that $D^k f = f^{(k)} \in C(\iota)$ (here D denotes differentiation). The set of all $f \in C(\iota)$ having all derivatives on ι is denoted by $C^\infty(\iota)$, and the set of those $f \in C^\infty(\iota)$ which vanish in a neighborhood of a and of b is denoted by $C_0^\infty(\iota)$.

The set of all $m \times 1$ matrix-valued functions $f\colon \iota \to \mathbb{C}^m$ (more precisely, equivalence classes of such functions) for which

$$\int_J |f|^p < \infty, \qquad |f|^2 = f^* f,$$

for every compact subinterval $J \subset \iota$ is denoted by $L^p_{\mathrm{loc}}(\iota)$, ($p \geqslant 1$), with $L^1_{\mathrm{loc}}(\iota) = L_{\mathrm{loc}}(\iota)$. The set of all $f \in L^p_{\mathrm{loc}}(\iota)$ which have compact support in ι is denoted by $L_0{}^p(\iota)$, whereas the set of all $f \in L^p_{\mathrm{loc}}(\iota)$ for which

$$\int_\iota |f|^p < \infty$$

is denoted by $L^p(\iota)$.

Let $AC^0_{\mathrm{loc}}(\iota) = AC_{\mathrm{loc}}(\iota)$ be the set of all $m \times 1$ matrix-valued functions $f\colon \iota \to \mathbb{C}^m$ whose components are absolutely continuous on each compact subinterval of ι, and, for $k \in \mathbb{N}$, let $AC^k_{\mathrm{loc}}(\iota)$ be the set of all those $f \in C^k(\iota)$ such that $f^{(k)} \in AC_{\mathrm{loc}}(\iota)$.

If F, G are matrix-valued functions on ι, having the same number of rows, we set

$$(F, G)_2 = \int_\iota G^* F, \qquad (F, G)_{2,J} = \int_J G^* F,$$

provided that the components of G^*F are in $L(\iota)$ and $L(J)$, respectively, where J is a subinterval of ι. We emphasize that F on G need not have columns in $L^2(\iota)$.

All our regularity results depend on the following simple lemma, whose proof we include for completeness.

Lemma 2.1. *If $f \in L_{\mathrm{loc}}(\iota)$, and if for some $k \in \mathbb{N}$ we have $(f, \varphi^{(k)})_2 = 0$ for all $\varphi \in C_0{}^\infty(\iota)$, then f is (after correction on a set of measure 0) a polynomial of degree at most $k - 1$.*

Proof. Let us first consider the case $k = 1$. Let $\alpha\colon \iota \to \mathbb{R}$ be in $C_0{}^\infty(\iota)$ and satisfy $\int_\iota \alpha = 1$. For $\varphi \in C_0{}^\infty(\iota)$ define

$$\psi = \varphi - \alpha \left(\int_\iota \varphi \right) = \varphi - \alpha(\varphi, I_m)_2;$$

then $\psi \in C_0{}^\infty(\iota)$ and $\int_\iota \psi = 0_m{}^1$. This implies that the function Ψ, defined by

$$\Psi(x) = \int_a^x \psi(t)\, dt,$$

is in $C_0{}^\infty(\iota)$, so that $(f, \Psi')_2 = (f, \psi)_2 = 0$. This is readily seen to imply $(f, \varphi)_2 = (I_m, \varphi)_2 (f, \alpha I_m)_2 = ((f, \alpha I_m)_2, \varphi)_2$, which means that

$$f = (f, \alpha I_m)_2 = \int_\iota \alpha f, \qquad \text{a.e. (almost everywhere).}$$

Now suppose the lemma is true for some k. Let $\varphi \in C_0^\infty(\iota)$ and decompose this as before. We then obtain

$$(f, \varphi^{(k)})_2 = (f, \psi^{(k)})_2 + (I_m, \varphi)_2(f, \alpha^{(k)}I_m)_2 .$$

If we assume $(f, \varphi^{(k+1)})_2 = 0$ for all $\varphi \in C_0^\infty(\iota)$, then $(f, \psi^{(k)})_2 = 0$, since $\psi^{(k)} = \Psi^{(k+1)}$, where $\Psi \in C_0^\infty(\iota)$. In addition, we obtain by partial integration

$$(I_m, \varphi)_2 = \int_\iota \frac{(-1)^k}{k!} x^k(\varphi^{(k)}(x))^* \, dx.$$

The induction assumption then tells us that a.e.,

$$f - \frac{(-1)^k}{k!} x^k(f, \alpha^{(k)}I_m)_2$$

is a polynomial of degree at most $k - 1$, or that f is, a.e., a polynomial of degree at most k. This proves the lemma.

On the open interval ι we now consider a differential expression L of order $n \geqslant 0$,

$$L = \sum_{k=0}^{n} P_k D^k,$$

where the P_k are $m \times m$ matrix-valued functions whose components are complex valued, measurable, and locally bounded functions on ι. If some component of P_n does not vanish on some set of positive measure, we shall say that the order of L is n. The Lagrange adjoint L^+ of L is given by

$$L^+ = \sum_{k=0}^{n} (-1)^k D^k P_k^* .$$

Associated with L we inductively define the *L-pseudo-derivatives* $f_L^{\{i\}}$, $0 \leqslant i \leqslant n$, of an $m \times 1$ matrix-valued function $f: \iota \to \mathbb{C}^m$ as follows: $f_L^{\{0\}} = (-1)^n P_n^* f$. If $f_L^{\{0\}} \in AC_{\mathrm{loc}}(\iota)$, then $f_L^{\{1\}} = Df_L^{\{0\}} + (-1)^{n-1}P_{n-1}^* f$, so that $f_L^{\{1\}}$ is defined a.e. If after correction on a null set $f_L^{\{i\}} \in AC_{\mathrm{loc}}(\iota)$, then

$$f_L^{\{i+1\}} = Df_L^{\{i\}} + (-1)^{n-i-1}P_{n-i-1}^* f.$$

THEOREM 2.1 (Generalized Green's Formula). *Suppose $f \in AC_{\mathrm{loc}}^{n-1}(\iota)$ and $g \in L_{\mathrm{loc}}(\iota)$, with $g_L^{\{i\}} \in AC_{\mathrm{loc}}(\iota)$, $i = 0, \ldots, n - 1$. Then for any $a_0, b_0 \in \iota$*

$$\int_{a_0}^{b_0} g^*(Lf) - (g_L^{\{n\}})^* f = \sum_{j=1}^{n} (-1)^j (g_L^{\{n-j\}})^* f^{(j-1)} \Big|_{a_0}^{b_0}$$

Proof. We find by induction that for $i = 0, \ldots, n-1$ the left side is equal to

$$\int_{a_0}^{b_0} \sum_{k=i+1}^{n} g^* P_k D^k f + \int_{a_0}^{b_0} (-1)^{i+1} D(g_L^{\{n-i-1\}})^* D^i f$$
$$+ \sum_{j=1}^{i} (-1)^j (g_L^{\{n-j\}})^* f^{(j-1)} \Big|_{a_0}^{b_0},$$

where for $i = 0$ the last term in the latter sum is absent. An integration by parts of the identity for $i = n - 1$ gives the desired result.

LEMMA 2.2. *Suppose $P_k \in C^k(\iota)$ (i.e., the columns of P_k are in $C^k(\iota)$) for $k = 0, \ldots, n$. If $f \in AC_{\mathrm{loc}}^{s-1}(\iota)$ for some s, $1 \leqslant s \leqslant n$, then (after correction on a set of measure 0) $f_L^{\{i\}} \in AC_{\mathrm{loc}}(\iota)$ for $i = 0, \ldots, s-1$. In particular, if $f \in AC_{\mathrm{loc}}^{n-1}(\iota)$ then*

$$f_L^{\{n\}} = \sum_{j=0}^{n} (-1)^j D^j(P_j^* f) = L^+ f, \qquad \text{a.e.}$$

Proof. It is clear that $f_L^{\{0\}} \in AC_{\mathrm{loc}}(\iota)$, and by definition

$$f_L^{\{1\}} = D f_L^{\{0\}} + (-1)^{n-1} P_{n-1}^* f \qquad \text{a.e.},$$

and thus

$$f_L^{\{1\}} = (-1)^n D(P_n^* f) + (-1)^{n-1} P_{n-1}^* f \quad \text{a.e.} \tag{2.1}$$

In case $s > 1$ the right side is in $AC_{\mathrm{loc}}(\iota)$, which implies that $f_L^{\{1\}} \in AC_{\mathrm{loc}}(\iota)$ after correction on a set of measure 0. An induction shows that

$$f_L^{\{i\}} = \sum_{j=0}^{i} (-1)^{n-i+j} D^j(P_{n-i+j}^* f) \qquad \text{a.e., } i = 0, \ldots, s, \tag{2.2}$$

with equality everywhere for $i = 0, \ldots, s-1$.

LEMMA 2.3. *Suppose $P_k \in C^k(\iota)$, $k = 0, \ldots, n$, and $P_n(x)$ is invertible for $x \in \iota$. If (after correction on a set of measure 0) $f_L^{\{i\}} \in AC_{\mathrm{loc}}(\iota)$ for $i = 0, \ldots, s-1$, for some s, $1 \leqslant s \leqslant n$, then (after correction on a set of measure 0) $f \in AC_{\mathrm{loc}}^{s-1}(\iota)$.*

Proof. Since $f_L^{\{0\}} \in AC_{\mathrm{loc}}(\iota)$, we have

$$f = (-1)^n (P_n^*)^{-1} f_L^{\{0\}} \in AC_{\mathrm{loc}}(\iota),$$

for the columns of $(P_n^*)^{-1}$ are in $AC_{\mathrm{loc}}(\iota)$. Moreover we have relation (2.1). In case $s > 1$ we see that f' is equal a.e. to a function in $AC_{\mathrm{loc}}(\iota)$, and consequently $f' \in AC_{\mathrm{loc}}(\iota)$, which implies $f \in AC_{\mathrm{loc}}^1(\iota)$. In general, for $i = 0, \ldots, s-1$ we have

$$f_L^{\{i\}} = \sum_{j=0}^{i} (-1)^{n-i+j} D^j(P_{n-i+j}^* f),$$

and hence

$$(-1)^n P_n^* D^i f = f_L^{\{i\}} - \sum_{k=0}^{i-1} \left[\sum_{j=k}^{i} (-1)^{n-i+j} \binom{j}{k} D^{j-k} P_{n-i+j}^* \right] D^k f,$$

so that $D^i f \in AC_{\text{loc}}(\iota)$, and hence $f \in AC_{\text{loc}}^{s-1}(\iota)$.

Remark. If we let $f \in AC_{\text{loc}}^{n-1}(\iota)$ in Lemma 2.2, then we have equality everywhere in (2.2) for $i = 0,..., n - 1$ and equality a.e. for $i = n$. In this last case we find the familiar fact that if

$$L^+ f = \sum_{k=0}^{n} Q_k D^k f,$$

then

$$Q_k = \sum_{j=k}^{n} (-1)^j \binom{j}{k} D^{j-k} P_j^*, \qquad k = 0,..., n.$$

Using the equality everywhere for $i = 0,..., n - 1$ we find the usual Green's formula

$$\int_{a_0}^{b_0} g^*(Lf) - (L^+g)^* f = \sum_{j=1}^{n} \sum_{k=0}^{n-j} (-1)^k (g^* P_{j+k})^{(k)} f^{(j-1)} \Big|_{a_0}^{b_0},$$

for $f, g \in AC_{\text{loc}}^{n-1}(\iota)$ and $a_0, b_0 \in \iota$.

Now consider a second differential expression

$$M = \sum_{k=0}^{\nu} Q_k D^k,$$

where the Q_k are $m \times m$ matrix-valued functions whose elements are complex-valued, measurable, and locally bounded functions on ι. Associated with M, we have the M-pseudo-derivatives $f_M^{\{i\}}$, $0 \leqslant i \leqslant \nu$, of an $m \times 1$ matrix-valued function $f: \iota \to \mathbb{C}^m$.

THEOREM 2.2. *Let $f, g \in L_{\text{loc}}(\iota)$ be such that*

$$(f, L\varphi)_2 = (g, M\varphi)_2, \qquad \varphi \in C_0^\infty(\iota), \tag{2.3}$$

and suppose $n > \nu$. Then (after correction on a set of measure 0) $f_L^{\{i\}} \in AC_{\text{loc}}(\iota)$, $i = 0,..., n - \nu - 1$.

If, in addition, $g_M^{\{i\}} \in AC_{\text{loc}}(\iota)$ for $i = 0,..., s$ for some s, $0 \leqslant s \leqslant \nu - 1$, then (after correction on a set of measure 0) $f_L^{\{i\}} \in AC_{\text{loc}}(\iota)$, $i = 0,..., n - \nu + s$. In case $s = \nu - 1$ we have

$$f_L^{\{n\}} = g_M^{\{\nu\}} \qquad \text{a.e.} \tag{2.4}$$

If $P_k \in C^k(\iota)$, $k = 0,\dots, n$, $Q_k \in C^k(\iota)$, $k = 0,\dots, \nu$, *and* $P_n(x)$, $Q_\nu(x)$ *are invertible for* $x \in \iota$, *then* (2.4) *implies* $f \in AC_{\mathrm{loc}}^{n-1}(\iota)$, $g \in AC_{\mathrm{loc}}^{\nu-1}(\iota)$ *and*

$$L^+ f = M^+ g \qquad a.e.$$

Proof. From (2.3) we have

$$\int_\iota \sum_{j=0}^{n} (D^j \varphi^*) P_j^* f = \int_\iota \sum_{j=0}^{\nu} (D^j \varphi^*) Q_j^* g. \tag{2.5}$$

Let $c \in \iota$ and introduce the following functions defined for $x \in \iota$:

$$F_{j,0}^L(x) = P_j^*(x) f(x), \qquad j = 0,\dots, n,$$

$$F_{j,k}^L(x) = \int_c^x F_{j,k-1}^L, \qquad j = 0,\dots, n,\ k = 1,\dots, n-j,$$

and

$$G_{j,0}^M(x) = Q_j^*(x) g(x), \qquad j = 0,\dots, \nu,$$

$$G_{j,k}^M(x) = \int_c^x G_{j,k-1}^M, \qquad j = 0,\dots, \nu,\ k = 1,\dots, n-j.$$

Integration by parts in (2.5) then yields

$$\int_\iota \sum_{j=0}^{n} (-1)^j (D^n \varphi^*) F_{n-j,j}^L = \int_\iota \sum_{j=0}^{\nu} (-1)^{j+n-\nu} (D^n \varphi^*)\, G_{\nu-j,j+n-\nu}^M.$$

Hence by Lemma 2.1 we obtain

$$\sum_{j=0}^{n} (-1)^j F_{n-j,j}^L = \sum_{j=0}^{\nu} (-1)^{j+n-\nu} G_{\nu-j,j+n-\nu}^M + R_{n-1} \qquad \text{a.e.},$$

where R_{n-1} is a polynomial of degree at most $n-1$. By induction for $i = 0,\dots, n-\nu-1$ we can now prove that (after correction on a set of measure 0) $f_L^{\{i\}} \in AC_{\mathrm{loc}}(\iota)$ and that

$$(-1)^n f_L^{\{i\}} = -\sum_{j=i+1}^{n} (-1)^j F_{n-j,j-i}^L + \sum_{j=0}^{\nu} (-1)^{j+n-\nu} G_{\nu-j,j+n-\nu-i}^M + D^i R_{n-1}$$

for $i = 0,\dots, n-\nu$. In case $i = n-\nu$ we may write

$$(-1)^n f_L^{\{i\}} = -\sum_{j=i+1}^{n} (-1)^j F_{n-j,j-i}^L + \sum_{j=1}^{\nu} (-1)^{j+n-\nu} G_{\nu-j,j+n-\nu-i}^M$$
$$+ D^i R_{n-1} + (-1)^{n-\nu} G_{\nu,0}^M.$$

Now the sums on the right side are in $AC_{\text{loc}}(\iota)$. So if $g_M^{\{0\}} = (-1)^\nu G_{\nu,0}^M \in AC_{\text{loc}}(\iota)$, it follows that $f_L^{\{n-\nu\}} \in AC_{\text{loc}}(\iota)$. We continue by induction. If $g_M^{\{i\}} \in AC_{\text{loc}}(\iota)$ for $i = 0,\ldots, s$ $(0 \leqslant s \leqslant \nu - 1)$, then for $i = n - \nu,\ldots, n - \nu + s$ we have $f_L^{\{i\}} \in AC_{\text{loc}}(\iota)$ and

$$(-1)^n f_L^{\{i\}} = - \sum_{j=i+1}^{n} (-1)^j F_{n-j,j-i}^L + \sum_{j=i+1-n+\nu}^{\nu} (-1)^{j+n-\nu} G_{\nu-j,j+n-\nu-i}^M + D^i R_{n-1} + (-1)^n g_M^{\{i+\nu-n\}}.$$

In particular, if $s = \nu - 1$, and we consider the above relation for $i = n - 1$, we obtain

$$f_L^{\{n-1\}} = -F_{0,1}^L + G_{0,1}^M + g_M^{\{\nu-1\}} + (-1)^n D^{n-1} R_{n-1},$$

which gives

$$f_L^{\{n\}} = g_M^{\{\nu\}} \qquad \text{a.e.}$$

The rest of the statements follow from Lemmas 2.2 and 2.3.

COROLLARY. *Let $P_k \in C^k(\iota)$, $k = 0,\ldots, n$, and suppose $P_n(x)$ is invertible for $x \in \iota$. If $f, g \in L_{\text{loc}}(\iota)$ are such that*

$$(f, L\varphi)_2 = (g, \varphi)_2, \qquad \varphi \in C_0^\infty(\iota),$$

then $f \in AC_{\text{loc}}^{n-1}(\iota)$ and $L^+f = g$ a.e. *If $g = 0_m{}^1$ a.e., i.e., we have*

$$(f, L\varphi)_2 = 0, \qquad \varphi \in C_0^\infty(\iota),$$

then $f \in C^n(\iota)$ and $L^+f = 0_m{}^1$.

We consider in more detail the case when L has coefficients $P_k \in C^k(\iota)$, $k = 0,\ldots, n$, and $P_n(x)$ is invertible for all $x \in \iota$. In $L^2(\iota)$ we define the operator L_0 to be L restricted to $C_0^\infty(\iota)$. Thus, in terms of graphs

$$L_0 = \{\{\varphi, L\varphi\} \mid \varphi \in C_0^\infty(\iota)\}.$$

Its adjoint L_0^* in $L^2(\iota)$ is just the set

$$L_0^* = \{\{f, g\} \in L^2(\iota) \oplus L^2(\iota) \mid (L\varphi, f)_2 - (\varphi, g)_2 = 0, \text{ all } \varphi \in C_0^\infty(\iota)\},$$

and we see from the corollary to Theorem 2.2 that if $\{f, g\} \in L_0^*$ then $f \in AC_{\text{loc}}^{n-1}(\iota)$, $g = L^+f$. Thus we define the maximal operator $L_{\max}^+$ for L^+ in $L^2(\iota)$ as

$$L_{\max}^+ = \{\{f, L^+f\} \mid f \in C^{n-1}(\iota) \cap L^2(\iota), f^{(n-1)} \in AC_{\text{loc}}(\iota), L^+f \in L^2(\iota)\},$$

and we have $L_0^* \subset L_{\max}^+$. But Green's formula shows that

$$(L\varphi, f)_2 - (\varphi, L^+ f)_2 = 0, \qquad \varphi \in C_0^\infty(\iota), f \in \mathfrak{D}(L_{\max}^+),$$

and hence $L_{\max}^+ \subset L_0^*$, or $L_0^* = L_{\max}^+$. If we define L_0^+ to be L^+ restricted to $C_0^\infty(\iota)$, we have $(L_0^+)^* = L_{\max}$, the maximal operator for L in $L^2(\iota)$,

$$L_{\max} = \{\{f, Lf\} \mid f \in C^{n-1}(\iota) \cap L^2(\iota), f^{(n-1)} \in AC_{\mathrm{loc}}(\iota), Lf \in L^2(\iota)\}.$$

The minimal operators $L_{\min}$, $L_{\min}^+$, for L and L^+ in $L^2(\iota)$ are defined by

$$L_{\min} = (L_0)^c, \qquad L_{\min}^+ = (L_0^+)^c.$$

Then we see that $L_{\min}$, $L_{\min}^+$, $L_{\max}$, $L_{\max}^+$ are all closed operators, and

$$L_0 \subset L_{\min} = (L_{\max}^+)^* \subset L_{\max} = (L_{\min}^+)^* = (L_0^+)^*,$$

$$L_0^+ \subset L_{\min}^+ = (L_{\max})^* \subset L_{\max}^+ = (L_{\min})^* = (L_0)^*.$$

For $f \in \mathfrak{D}(L_{\max})$, $f^+ \in \mathfrak{D}(L_{\max}^+)$, we may write Green's formula as

$$\int_{a_0}^{b_0} (f^+)^* Lf - (L^+ f^+)^* f = [f, f^+](b_0) - [f, f^+](a_0),$$

where $[a_0, b_0] \subset \iota = (a, b)$, and

$$[f, f^+](x) = (\tilde{f}^+)^*(x) \mathscr{B}_L(x) \tilde{f}(x).$$

Here we are using the notation $\tilde{u}$, for any $j \times k$ matrix-valued function u on ι having $n-1$ derivatives, to mean the $nj \times k$ matrix-valued function on ι given by

$$\tilde{u} = \begin{pmatrix} u \\ u' \\ \vdots \\ u^{(n-1)} \end{pmatrix}.$$

Thus $\tilde{f}$ is $nm \times 1$. As for $\mathscr{B}_L$, it is a continuous invertible $mn \times mn$ matrix-valued function on ι depending only on the coefficients P_k of L and their derivatives. If $\mathscr{B}_L = (\mathscr{B}_{Ljk})$, $j, k = 1,..., n$, is considered as an $n \times n$ matrix whose entries are $m \times m$ matrices $\mathscr{B}_{Ljk}$, then

$$\begin{aligned} \mathscr{B}_{Ljk} &= \sum_{p=j+k-1}^{n} (-1)^{p-k} \binom{p-k}{j-1} P_p^{(p-j-k+1)}, \qquad j+k \leqslant n+1, \\ &= 0_m{}^m, \qquad j+k > n+1. \end{aligned}$$

In particular $\mathscr{B}_{Ljk} = (-1)^{n-k}P_n$ if $j + k = n + 1$, and this implies $\det \mathscr{B}_L(x) = (\det P_n(x))^n \neq 0$, $x \in \iota$. It follows from Green's formula that the limits

$$\lim_{x \to a}[f, f^+](x) = [f, f^+](a), \qquad \lim_{x \to b}[f, f^+](x) = [f, f^+](b),$$

exist for all $f \in \mathfrak{D}(L_{\max})$, $f^+ \in \mathfrak{D}(L_{\max}^+)$.

We say that L is *regular at the endpoint a* if a is finite and the coefficients P_k of L can be extended to $[a, b)$ so that $P_k \in C^k([a, b))$, $k = 0, 1, \ldots, n$, and $P_n(x)$ is invertible for all $x \in [a, b)$; similarly for the endpoint b. The differential expression L is *regular* on ι if it is regular at a and b. If L is regular at a then so is L^+, and, in this case, for all $f \in \mathfrak{D}(L_{\max}) \cup \mathfrak{D}(L_{\max}^+)$ we have $\tilde{f}(x)$ tends to a finite limit $\tilde{f}(a)$ as $x \to a$. To see this we note that $[f, f^+](x) = (\tilde{f}^+)^*(x) \mathscr{B}_L(x) \tilde{f}(x)$ tends to a limit as $x \to a$ for all $f \in \mathfrak{D}(L_{\max})$, $f^+ \in \mathfrak{D}(L_{\max}^+)$, and we have $(\mathscr{B}_L(x))^{-1}$ tends to a limit $(\mathscr{B}_L(a))^{-1}$, since L is regular at a. Thus it is sufficient to show $\mathscr{B}_L(x)\tilde{f}(x)$ (or $(\tilde{f}^+)^*(x)\mathscr{B}_L(x)$) tends to a limit. Let

$$v_k(x) = \frac{(x-a)^{k-1}}{(k-1)!} I_m, \qquad a \leqslant x < b,\ k = 1, \ldots, n,$$

and put $f_k^+ = \varphi v_k$, where $\varphi \in C^\infty([a, b))$ is such that $\varphi(x) = 1$ on some interval $[a, a+\epsilon] \subset [a, b)$, $\epsilon > 0$, and $\varphi(x) = 0$ for x near b. Thus $f_k^+ \in \mathfrak{D}(L_{\max}) \cap \mathfrak{D}(L_{\max}^+)$, and $\tilde{f}_k^+(x) \to \tilde{v}_k(a)$ as $x \to a$, where $\tilde{v}_k(a)$ is an $nm \times 1$ matrix whose entries are $0_m{}^1$ except the kth which is I_m. If we let $f^+ = (f_1^+, \ldots, f_n^+)$, we see that $\tilde{f}^+(x) \to I_{nm}$ as $x \to a$, so that $\mathscr{B}_L(x)\tilde{f}(x)$ tends to a limit as $x \to a$, for all $f \in \mathfrak{D}(L_{\max})$. We now see that $\mathfrak{D}(L_{\max}) \cup \mathfrak{D}(L_{\max}^+)$ can be considered as a subset of $C^{n-1}([a, b))$. Since

$$f^{(n)} = P_n^{-1}[Lf - (P_{n-1}f^{(n-1)} + \cdots + P_0 f)],$$

and $Lf \in L^2(\iota)$ for $f \in \mathfrak{D}(L_{\max})$, we have that $f^{(n)} \in L^2([a, a+\epsilon])$ for $\epsilon > 0$ such that $[a, a+\epsilon] \subset [a, b)$, and so $f^{(n)} \in L^1([a, a+\epsilon])$ for such ϵ. Hence $f^{(n-1)} \in AC([a, a+\epsilon))$ for all such ϵ, or, as we shall say $f^{(n-1)} \in AC_{\mathrm{loc}}([a, b))$. We have shown that if L is regular at a then

$$L_{\max} = \{\{f, Lf\} \mid f \in C^{n-1}([a, b)) \cap L^2(\iota), f^{(n-1)} \in AC_{\mathrm{loc}}([a, b)), Lf \in L^2(\iota)\},$$
$$L_{\max}^+ = \{\{f, Lf\} \mid f \in C^{n-1}([a, b)) \cap L^2(\iota), f^{(n-1)} \in AC_{\mathrm{loc}}([a, b)), L^+f \in L^2(\iota)\}.$$

Note that the essential fact shown in the proof above is that the map κ: $\mathfrak{D}(L_{\max}) \cap \mathfrak{D}(L_{\max}^+) \to \mathbb{C}^{nm}$ given by

$$\kappa(f^+) = \tilde{f}^+(a) = \lim_{x \to a} \tilde{f}^+(x), \qquad f^+ \in \mathfrak{D}(L_{\max}) \cap \mathfrak{D}(L_{\max}^+),$$

is surjective.

If L is regular at a then $f \in \mathfrak{D}(L_{\min}) \cup \mathfrak{D}(L^+_{\min})$ implies that $\tilde{f}(a) = 0^1_{nm}$. If, for example, $f \in \mathfrak{D}(L_{\min})$ then

$$(Lf, f^+)_2 - (f, L^+f^+)_2 = [f, f^+](b) - [f, f^+](a) = 0 \tag{2.6}$$

for all $f^+ \in \mathfrak{D}(L^+_{\max})$, since $L_{\min} = (L^+_{\max})^*$. Letting $f^+ = (f_1{}^+,..., f_n{}^+)$ we find that $[f, f^+](b) = 0^1_{nm}$ and

$$[f, f^+](a) = \mathscr{B}_L(a)\tilde{f}(a) = 0^1_{nm},$$

so that $\tilde{f}(a) = 0^1_{nm}$. In particular if L is regular on ι and $f \in \mathfrak{D}(L_{\min})$ then $f \in \mathfrak{D}(L_{\max})$ and $\tilde{f}(a) = \tilde{f}(b) = 0^1_{nm}$. Conversely, if $f \in \mathfrak{D}(L_{\max})$ and $\tilde{f}(a) = \tilde{f}(b) = 0^1_{nm}$, then (2.6) is valid for all $f^+ \in \mathfrak{D}(L^+_{\max})$ and $f \in \mathfrak{D}(L_{\min})$. Therefore we have, in case L is regular on ι,

$$L_{\min} = \{\{f, Lf\} \in L_{\max} \mid \tilde{f}(a) = \tilde{f}(b) = 0^1_{nm}\},$$
$$L^+_{\min} = \{\{f, L^+f\} \in L^+_{\max} \mid \tilde{f}(a) = \tilde{f}(b) = 0^1_{nm}\}.$$

We shall be considering a formally symmetric differential expression on $\iota = (a, b)$,

$$M = \sum_{k=0}^{\nu} Q_k D^k = M^+,$$

where $Q_k(m \times m) \in C^k(\iota)$, $k = 0,..., \nu$, and $Q_\nu(x)$ is invertible for all $x \in \iota$. Every such M can be written as

$$M = \sum_{j=0}^{\mu} (-1)^j D^j A_j D^j + i \sum_{j=0}^{\mu-1} (D^{j+1} B_j D^j + D^j B_j D^{j+1}), \qquad \nu = 2\mu, \tag{2.7}$$

or as

$$M = \sum_{j=0}^{\mu} (-1)^j D^j A_j D^j + i \sum_{j=0}^{\mu} (D^{j+1} B_j D^j + D^j B_j D^{j+1}), \qquad \nu = 2\mu + 1, \tag{2.8}$$

where the A_j, B_j are Hermitian $m \times m$ matrix-valued functions on ι such that $A_j \in C^{2j}(\iota)$, $j = 0,..., \mu$, and $B_j \in C^{2j+1}(\iota)$, $j = 0,..., \mu - 1$, in case $\nu = 2\mu$, or $j = 0,..., \mu$ in case $\nu = 2\mu + 1$. The proof for the scalar case $m = 1$ can be used. Consider (2.7), for example. We must have $Q^*_{2\mu} = Q_{2\mu}$ and $M_1 = M - D^\mu Q_{2\mu} D^\mu$ is again a formally symmetric differential expression of order at most $2\mu - 1$. Choose $A_\mu = (-1)^\mu Q_{2\mu}$. If M_1 has order $2\mu - 1$ with the leading coefficient $R_{2\mu-1}$, then $R^*_{2\mu-1} = -R_{2\mu-1}$, or, if we let $B_{\mu-1} = -(i/2)R_{2\mu-1}$, $B^*_{\mu-1} = B_{\mu-1}$. Then

$$M_1 - i(D^\mu B_{\mu-1} D^{\mu-1} + D^{\mu-1} B_{\mu-1} D^\mu)$$

is formally symmetric of order at most $2\mu - 2$. An induction now yields (2.7), and similarly for (2.8). We remark that the decompositions (2.7) and (2.8) of M are unique, i.e., the Hermitian matrices A_j, B_j are uniquely determined by M.

Our principal concern will be with even order, $\nu = 2\mu$, formally symmetric M. The study of such M is simplified by the introduction of *quasi-derivatives.* We define the matrix-valued functions Q_{jk} by

$$\begin{aligned} Q_{jj} &= A_j, & j &= 0,\ldots,\mu, \\ Q_{j+1\,j} &= i(-1)^{j+1}B_j, & j &= 0,\ldots,\mu-1, \\ Q_{j\,j+1} &= i(-1)^{j}B_j, & j &= 0,\ldots,\mu-1, \end{aligned}$$

and all other $Q_{jk} = 0_m{}^m$. Then $Q_{jk}^* = Q_{kj}$ and we may write M as

$$M = \sum_{j=0}^{\mu} \sum_{k=j+1}^{j+1} (-1)^j D^j Q_{jk} D^k.$$

In such sums we assume the terms which have $Q_{jk} = 0_m{}^m$ are zero, even though D^k is not defined for $k < 0$. For an $m \times 1$ matrix-valued function $f\colon \iota \to \mathbb{C}^m$ we introduce the *quasi-derivatives* of f with respect to M:

$$\begin{aligned} f^{[j]} &= D^j f, & j &= 0,\ldots,\mu-1, \\ f^{[\mu]} &= Q_{\mu\,\mu-1} D^{\mu-1} f + Q_{\mu\mu} D^{\mu} f, & & \qquad (2.9) \\ f^{[\mu+1]} &= \sum_{k=\mu-j-1}^{\mu-j+1} Q_{\mu-j\;k} D^k f - D f^{[\mu+j-1]}, & j &= 1,\ldots,\mu. \end{aligned}$$

The introduction of these for differential expressions of arbitrary order with complex coefficients goes back at least to Šin [26]. From the last line in (2.9) we obtain by induction

$$f^{[\nu]} = \sum_{j=0}^{s} \sum_{k=j-1}^{j+1} (-1)^j D^j Q_{jk} D^k f + (-1)^{s+1} D^{s+1} f^{[\nu-s-1]}, \qquad 0 \leqslant s \leqslant \mu - 1.$$

In particular, for $s = \mu - 1$ this shows that

$$f^{[\nu]} = \sum_{j=0}^{\mu} \sum_{k=j-1}^{j+1} (-1)^j D^j Q_{jk} D^k f = Mf.$$

Another relation that follows from the last line in (2.9) is

$$D f^{[\nu-i-1]} = \sum_{k=i-1}^{i+1} Q_{ik} D^k f - f^{[\nu-i]}, \qquad i = 0,\ldots,\mu-1.$$

THEOREM 2.3. *If $f \in AC_{\mathrm{loc}}^{\nu-1}(\iota)$ and $g \in AC_{\mathrm{loc}}^{\mu-1}(\iota)$, then*

$$g^*(Mf) = \sum_{i=0}^{\mu} \sum_{k=i-1}^{i+1} (D^i g^*) Q_{ik}(D^k f) - D\left\{\sum_{i=0}^{\mu-1} (g^{[i]})^* f^{[\nu-i-1]}\right\}.$$

Proof. We have

$$D\left\{\sum_{i=0}^{\mu-1} (g^{[i]})^* f^{[\nu-i-1]}\right\} = \sum_{i=0}^{\mu-1} \{(g^{[i]})^* D f^{[\nu-i-1]} + D(g^{[i]})^* f^{[\nu-i-1]}\}.$$

Using the above relations for quasi-derivatives we find

$$\begin{aligned}\sum_{i=0}^{\mu-1} (g^{[i]})^* D f^{[\nu-i-1]} &= \sum_{i=0}^{\mu-1} (g^{[i]})^* \left\{\sum_{k=i-1}^{i+1} Q_{ik} D^k f - f^{[\nu-i]}\right\} \\ &= \sum_{i=0}^{\mu-1} (g^{[i]})^* \left\{\sum_{k=i-1}^{i+1} Q_{ik} D^k f\right\} \\ &\quad - \sum_{i=1}^{\mu-1} (g^{[i]})^* f^{[\nu-i]} - (g^{[0]})^* f^{[\nu]},\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\sum_{i=0}^{\mu-1} D(g^{[i]})^* f^{[\nu-i-1]} &= \sum_{i=0}^{\mu-2} (g^{[i+1]})^* f^{[\nu-i-1]} + (D^\mu g^*) f^{[\mu]} \\ &= \sum_{i=1}^{\mu-1} (g^{[i]})^* f^{[\nu-i]} + \sum_{k=\mu-1}^{\mu+1} (D^\mu g^*) Q_{\mu k} D^k f.\end{aligned} \tag{2.11}$$

Now addition of (2.10) and (2.11) gives the desired result.

COROLLARY 1 (Lagrange identity in terms of quasi-derivatives). *If $f, g \in AC_{\mathrm{loc}}^{\nu-1}(\iota)$ then*

$$g^*(Mf) - (Mg)^* f = D\left\{\sum_{i=0}^{\mu-1} ((g^{[\nu-i-1]})^* f^{[i]} - (g^{[i]})^* f^{[\nu-i-1]})\right\}.$$

COROLLARY 2. *If $f \in \mathfrak{D}(M_{\max})$, $g \in AC^{\mu-1}(\iota)$, $g^{(\mu)} \in L^2(\iota)$, and $\bar{\iota}_0 = [a_0, b_0] \subset \iota = (a, b)$, then*

$$\begin{aligned}\int_{\iota_0} g^*(Mf) &= \int_{\iota_0} \sum_{i=0}^{\mu} \sum_{k=i-1}^{i+1} (D^i g^*) Q_{ik}(D^k f) \\ &\quad + (g_{[1]}^0)^* f_{[2]}^0,\end{aligned}$$

where $f_{[1]}^0$ is the one-column matrix with components $f(a_0), f^{[1]}(a_0), \ldots, f^{[\mu-1]}(a_0)$, $f(b_0), f^{[1]}(b_0), \ldots, f^{[\mu-1]}(b_0)$, and $f_{[2]}^0$ is the one-column matrix with elements

$$f^{[2\mu-1]}(a_0), \ldots, f^{[\mu]}(a_0), \; -f^{[2\mu-1]}(b_0), \ldots, -f^{[\mu]}(b_0).$$

COROLLARY 3 (Green's formula in terms of quasi-derivatives). *If* $f, g \in \mathfrak{D}(M_{\max})$,

$$(Mf, g)_{2,\bar{\iota}_0} - (f, Mg)_{2,\bar{\iota}_0} = (g^0_{[1]})^* f^0_{[2]} - (g^0_{[2]})^* f^0_{[1]} \,.$$

The integral

$$\int_{\iota_0} \sum_{i=0}^{\mu} \sum_{k=i-1}^{i+1} (D^i g^*) Q_{ik}(D^k f)$$

is called a "*Dirichlet integral*" on ι_0 associated with M.

3. Hilbert Spaces Associated with Positive Differential Operators

On $\iota = (a, b)$ we consider a differential expression of order $\nu \geqslant 0$,

$$M = \sum_{k=0}^{\nu} Q_k D^k,$$

where $Q_k(m \times m) \in C^k(\iota)$, $k = 0,..., \nu$, and $Q_\nu(x)$ is invertible for $x \in \iota$. On $C_0^\infty(\iota)$ we wish to associate an inner product (,) with M via

$$(\varphi, \psi) = (M\varphi, \psi)_2 \,, \qquad \varphi, \psi \in C_0^\infty(\iota).$$

Thus we must have, and we assume, M is formally symmetric, $M = M^+$, and nonnegative in the sense that

$$(M\varphi, \varphi)_2 \geqslant 0, \qquad \varphi \in C_0^\infty(\iota).$$

This in turn implies that the order of M must be even, $\nu = 2\mu$, and that the leading coefficient Q_ν has the property that $(-1)^\mu Q_\nu(x) > 0$, $x \in \iota$, in the sense that for each $x \in \iota$ there is a positive constant $c(x)$ such that

$$\xi^*(-1)^\mu Q_\nu(x)\xi \geqslant c(x)\, \xi^*\xi, \qquad \xi \in \mathbb{C}^m.$$

The usual proof in the scalar case carries over to the system case. If $\varphi \in C_0^\infty(\iota)$ let $\varphi_t(x) = e^{itx}\varphi(x)$, $t \in \mathbb{R}$, $x \in \iota$. Then $\varphi_t \in C_0^\infty(\iota)$ and

$$0 \leqslant (M\varphi_t \,, \varphi_t)_2 = a_\nu t^\nu + \cdots + a_0 \,, \qquad a_k \in \mathbb{R}, t \in \mathbb{R},$$

with $a_\nu = i^\nu(Q_\nu\varphi, \varphi)_2$. If $a_\nu \neq 0$, $\operatorname{sign}(M\varphi_t \,, \varphi_t)_2 = \operatorname{sign}(a_\nu t^\nu)$ for $|\, t \,| > t_0 > 0$, for some sufficiently large t_0. This readily implies $a_\nu \geqslant 0$ and $\nu = 2\mu$, an even integer. In turn this implies $i^\nu Q_\nu(x) = (-1)^\mu Q_\nu(x) > 0$, $x \in \iota$, since $Q_\nu(x)$ is invertible for $x \in \iota$.

We have seen in Section 2 that an $M = M^+$ of even order $\nu = 2\mu$ can be written as

$$M = \sum_{j=0}^{\mu} \sum_{k=j-1}^{j+1} (-1)^j D^j Q_{jk} D^k$$

for appropriate Q_{jk}, $Q_{jk}^* = Q_{kj}$, and therefore the inner product (φ, ψ) on $C_0^\infty(\iota)$ is what is often called a "*Dirichlet inner product*,"

$$(\varphi, \psi) = (M\varphi, \psi)_2 = \int_\iota \sum_{j=0}^{\mu} \sum_{k=j-1}^{j+1} (D^j\psi)^* Q_{jk}(D^k\varphi), \qquad \varphi, \psi \in C_0^\infty(\iota). \quad (3.1)$$

The Hilbert space $\mathfrak{H}_M$. Since we require

$$(\varphi, \varphi) = (\varphi, M\varphi)_2 = (M\varphi, \varphi)_2 \geqslant 0, \qquad \varphi \in C_0^\infty(\iota),$$

it is obvious that $\| \, \| = \sqrt{(\, , \,)}$ defines a seminorm on $C_0^\infty(\iota)$. Actually it is a norm there, as the following reasoning shows. Suppose $\varphi \in C_0^\infty(\iota)$ is such that $\| \varphi \| = 0$. Then

$$|(\varphi, \psi)| \leqslant \| \varphi \| \, \| \psi \| = 0$$

for all $\psi \in C_0^\infty(\iota)$. This and (3.1) imply that $(M\varphi, \psi)_2 = 0$ for all $\psi \in C_0^\infty(\iota)$, which yields $M\varphi = 0$, and hence $\varphi = 0$. We denote by $\mathfrak{H}_M$ the Hilbert space completion of $C_0^\infty(\iota)$ in this norm. This Hilbert space can be viewed as a subset of $L^2_{\text{loc}}(\iota)$ in certain cases, and we will guarantee this by making the following assumption for the remainder of this paper:

(3.2) ASSUMPTION. *For any compact subinterval $J \subset \iota$ there exists a positive constant $c(J)$ such that*

$$\| \varphi \| \geqslant c(J) \, \| \varphi \|_{2,J}, \qquad \varphi \in C_0^\infty(\iota).$$

A similar assumption has been made by Browder in his work on eigenfunction expansions for partial differential operators ([5, p. 371]; see also [6, p. 11]). A more complicated assumption has been made by Bennewitz in his study of pairs of differential operators [1, pp. 41, 42].

Let $\{f_n\}$ be a sequence of $C_0^\infty(\iota)$ elements for which $\| f_n - f_m \| \to 0$ as $n, m \to \infty$. This Cauchy sequence defines an element $F \in \mathfrak{H}_M$. However, on every compact subinterval $J \subset \iota$ we have $\| f_n - f_m \|_{2,J} \to 0$ by (3.2). Hence there exists an $f \in L^2_{\text{loc}}(\iota)$ such that $\| f_n - f \|_{2,J} \to 0$ on every compact subinterval $J \subset \iota$. For $\varphi \in C_0^\infty(\iota)$ we have by (3.1)

$$\begin{aligned} |(F, \varphi) - (f, M\varphi)_2| &= |(F - f_n, \varphi) + (f_n, \varphi) - (f, M\varphi)_2| \\ &= |(F - f_n, \varphi) + (f_n - f, M\varphi)_2| \leqslant \| F - f_n \| \, \| \varphi \| + \| f_n - f \|_{2,J} \| M\varphi \|_{2,J}, \end{aligned}$$

where J denotes the support of $M\varphi$. This shows that

$$(F, \varphi) = (f, M\varphi)_2, \qquad \varphi \in C_0^\infty(\iota).$$

This implies that the identity map on $C_0^\infty(\iota)$ has an extension $F \mapsto f$ of $\mathfrak{H}_M$ into $L^2_{\mathrm{loc}}(\iota)$, which is injective, so that we may, and do, identify $\mathfrak{H}_M$ as a subset of $L^2_{\mathrm{loc}}(\iota)$. Then we have

$$(f, \varphi) = (f, M\varphi)_2, \qquad f \in \mathfrak{H}_M, \ \varphi \in C_0^\infty(\iota), \tag{3.3}$$

and, since (3.2) extends to $\mathfrak{H}_M$,

$$\| f \| \geqslant c(J) \| f \|_{2,J}, \qquad f \in \mathfrak{H}_M. \tag{3.4}$$

A consequence of (3.3) is that if $f \in \mathfrak{H}_M$, $f \in C^\nu(\iota)$, and $Mf = 0$, then $f = 0$, for

$$(f, \varphi) = (f, M\varphi)_2 = (Mf, \varphi)_2 = 0, \qquad \varphi \in C_0^\infty(\iota),$$

implies $f = 0$.

Let M_F be the Friedrichs extension of M_0, the restriction of $M_{\max}$ to $C_0^\infty(\iota)$. We claim that $\mathfrak{D}(M_F)$ is in $\mathfrak{H}_M$. For if $f \in \mathfrak{D}(M_F)$ there exists a sequence $\varphi_n \in C_0^\infty(\iota)$ such that

$$\| f - \varphi_n \|_2 \to 0, \qquad \| \varphi_n - \varphi_m \| = (M(\varphi_n - \varphi_m), \varphi_n - \varphi_m)_2 \to 0,$$

and so $\| \varphi_n - \varphi_m \|_{2,J} \to 0$. If F is the element in $\mathfrak{H}_M$ determined by φ_n, then $\| F - \varphi_n \|_{2,J} \to 0$ for each compact subinterval $J \subset \iota$ and $\| f - \varphi_n \|_2 \to 0$ imply that $F = f$. Now for $f \in \mathfrak{D}(M_F)$ we have

$$\| f \|^2 = (f, f) = \lim (f, \varphi_n) = \lim (f, M\varphi_n)_2 = \lim (Mf, \varphi_n)_2 = (M_F f, f)_2,$$

and this shows that the inner product on $\mathfrak{D}(M_F) \subset \mathfrak{H}_M$ is

$$(f, g) = (Mf, g)_2, \qquad f, g \in \mathfrak{D}(M_F),$$

and also

$$(M_F f, f)_2 \geqslant c(J)^2 \| f \|_{2,J}^2, \qquad f \in \mathfrak{D}(M_F). \tag{3.5}$$

Therefore we see that $\mathfrak{H}_M$ can be viewed as the completion of $\mathfrak{D}(M_F)$ with the inner product $(M_F f, g)_2$, $f, g \in \mathfrak{D}(M_F)$.

We note that

$$C_0^\infty(\iota) \subset C_0^\nu(\iota) \subset \mathfrak{D}(M_{\min}) \subset \mathfrak{D}(M_F) \subset \mathfrak{H}_M. \tag{3.6}$$

THEOREM 3.1. *There exists a linear map $G_M : L_0^2(\iota) \to \mathfrak{H}_M$ with the following properties*:

(i) $(f, h)_2 = (f, G_M h), f \in \mathfrak{H}_M, h \in L_0^2(\iota)$,

(ii) $G_M M\varphi = \varphi, \varphi \in C_0^\infty(\iota)$,

(iii) $MG_M h = h, h \in L_0^2(\iota)$,

(iv) G_M *is injective and* $\mathfrak{R}(G_M)$ *is dense in* $\mathfrak{H}_M$.

Proof. Let $h \in L_0^2(\iota)$ with support $J \subset \iota$. A consequence of (3.4) is that

$$|(f, h)_2| \leqslant \|f\|_{2,J} \|h\|_{2,J} \leqslant d(J)\|f\|, \qquad f \in \mathfrak{H}_M,$$

where $d(J)$ is a positive constant depending on h. This means that for a fixed $h \in L_0^2(\iota)$ the map $f \mapsto (f, h)_2$ is a continuous linear functional on $\mathfrak{H}_M$. Hence there exists a unique element, say $G_M h$, in $\mathfrak{H}_M$ for which (i) holds. It is clear that this defines a linear map G_M of $L_0^2(\iota)$ into $\mathfrak{H}_M$. Next we note that $\varphi \in C_0^\infty(\iota)$ implies $M\varphi \in L_0^2(\iota)$, so that the left side of (ii) is defined. According to (3.3) and (i) we have

$$(f, \varphi) = (f, M\varphi)_2 = (f, G_M M\varphi), \qquad f \in \mathfrak{H}_M, \varphi \in C_0^\infty(\iota),$$

which proves (ii). As to (iii) note that

$$(\varphi, h)_2 = (\varphi, G_M h) = (M\varphi, G_M h)_2, \qquad \varphi \in C_0^\infty(\iota), h \in L_0^2(\iota).$$

According to the Corollary to Theorem 2.2 this implies that (after correction on a set of measure 0) $G_M h \in AC_{\mathrm{loc}}^{\nu-1}(\iota)$ (so the left side of (iii) is defined) and $MG_M h = h$ a.e. proving (iii). This also implies G_M is injective. If $f \in \mathfrak{H}_M \ominus \mathfrak{R}(G_M)$ we see from (i) that $(f, h)_2 = 0$ for all $h \in L_0^2(\iota)$, or $f = 0$, proving (iv).

We note that if $h \in C(\iota) \cap L_0^2(\iota)$ then $Gh \in C^\nu(\iota) \cap \mathfrak{H}_M$.

A global inequality for $\mathfrak{H}_M$. Let us now consider an assumption stronger than (3.2), which was made by Brauer [3] in his work on pairs of differential expressions:

(3.7) ASSUMPTION. *There exists a constant* $c > 0$ *such that*

$$\|\varphi\| \geqslant c\|\varphi\|_2, \qquad \varphi \in C_0^\infty(\iota).$$

Using an argument just like that following (3.2), we see that the identity map on $C_0^\infty(\iota)$ can be extended to an injection of $\mathfrak{H}_M$ into $L^2(\iota)$, so that we may, and do, identify $\mathfrak{H}_M$ as a subset of $L^2(\iota)$. We have

$$(f, \varphi) = (f, M\varphi)_2, \qquad f \in \mathfrak{H}_M, \varphi \in \mathfrak{D}(M_{\min}). \tag{3.8}$$

It is clear that (3.8) is valid for $\varphi \in C_0^\infty(\iota)$, and this implies (3.8) itself. The inequality in (3.7) extends to $\mathfrak{H}_M$,

$$\|f\| \geqslant c\|f\|_2, \qquad f \in \mathfrak{H}_M. \tag{3.9}$$

On account of the inequality

$$|(f, h)_2| \leqslant \|f\|_2 \|h\|_2 \leqslant (c^{-1} \|h\|_2) \|f\|, \qquad f \in \mathfrak{H}_M,\ h \in L^2(\iota),$$

the mapping G_M has an extension, call it G_M also, to an injection $G_M: L^2(\iota) \to \mathfrak{H}_M$ such that Theorem 3.1 is valid with $L_0^2(\iota)$ replaced by $L^2(\iota)$ everywhere. In fact assuming (3.7) we can identify G_M more precisely. In this case $(M_F)^{-1}$ exists as a bounded operator defined on all of $L^2(\iota)$, and it coincides with the operator G_M. To see this let $h \in L^2(\iota)$ and put $f = G_M h - (M_F)^{-1} h$. Then $f \in \mathfrak{H}_M$ and $Mf = 0$ (by Theorem 3.1), which implies $f = 0$ by the remark following (3.4). We note that $\mathfrak{H}_M$ is the domain $\mathfrak{D}((M_F)^{1/2})$ of the positive square root $(M_F)^{1/2}$ of M_F.

We give an example to show that (3.7) is strictly stronger than (3.2). Let $\iota = (0, \infty)$, $m = 1$, and $M = -D^2$. For $\varphi \in C_0^\infty(\iota)$ we have

$$\varphi(x) = \int_0^x \varphi'(t)\, dt,$$

which readily implies that, for every compact subinterval $J \subset \iota$,

$$\|\varphi\|_{2,J}^2 \leqslant \left(\int_J t\, dt\right) \|\varphi'\|_2^2,$$

which shows that (3.2) is valid. However, if we let for $n \geqslant 2$

$$\begin{aligned} \varphi_n(x) &= n^{-\gamma}(x-1)^3(n-x)^2, && 1 \leqslant x \leqslant n, \\ &= 0, && 0 < x < 1,\ x > n, \end{aligned}$$

where $11/2 < \gamma < 13/2$, then $\varphi_n \in C_0^2(\iota)$, so that $\varphi_n \in \mathfrak{H}_M$ on account of (3.6). A calculation shows that

$$\|\varphi_n\| \to 0, \qquad \|\varphi_n\|_2 \to +\infty,$$

which implies that (3.9) does not hold, and hence (3.7) cannot be valid.

The Hilbert space $\mathfrak{H}_S$. If M satisfies (3.2), then the operator M_0 in $L^2(\iota)$ may have symmetric extensions S in $L^2(\iota)$ which satisfy the following condition:

(3.10) *For each compact subinterval* $J \subset \iota$ *there exists a* $c(J) > 0$ *such that*

$$(Sf, f)_2 = (Mf, f)_2 \geqslant (c(J))^2 (f, f)_{2,J}, \qquad f \in \mathfrak{D}(S).$$

In this situation the completion of $\mathfrak{D}(S)$ with the inner product

$$(f, g) = (Sf, g)_2 = (f, Sg)_2, \qquad f, g \in \mathfrak{D}(S),$$

is a Hilbert space $\mathfrak{H}_S$ which includes $C_0^\infty(\iota)$. If $f_n \in \mathfrak{D}(S)$, $\|f_n - f_m\| \to 0$, then f_n determines an $F \in \mathfrak{H}_S$, $\|f_n - F\| \to 0$, and also, by (3.10), an $f \in L^2_{\mathrm{loc}}(\iota)$ such that $\|f_n - f\|_{2,J} \to 0$ for each compact subinterval $J \subset \iota$. The map $F \mapsto f$ is linear, and will be an injection if and only if S satisfies:

(3.11) $f_n \in \mathfrak{D}(S)$, $\|f_n - f_m\| \to 0$, $\|f_n\|_{2,J} \to 0$, *each compact subinterval* $J \subset \iota$, *implies* $\|f_n\| \to 0$.

Assuming (3.10) as well as (3.11), we see that F can be identified with f, $\mathfrak{H}_S \subset L^2_{\mathrm{loc}}(\iota)$. We have for $f \in \mathfrak{H}_S$, $\varphi \in C_0^\infty(\iota)$, $\|f - f_n\| \to 0$, $f_n \in \mathfrak{D}(S)$,

$$(f, \varphi) = \lim(f_n, \varphi) = \lim(f_n, M\varphi)_2 = (f, M\varphi)_2,$$

which shows that

$$(f, \varphi) = (f, M\varphi)_2, \qquad f \in \mathfrak{H}_S, \ \varphi \in C_0^\infty(\iota). \tag{3.12}$$

The inequality in (3.10) extends to $\mathfrak{H}_S$,

$$\|f\| \geqslant c(J)\|f\|_{2,J}, \qquad f \in \mathfrak{H}_S. \tag{3.13}$$

It is clear that the inner product of $\mathfrak{H}_S$, restricted to $C_0^\infty(\iota)$, is the same as the inner product for $\mathfrak{H}_M$. Thus the completion of $C_0^\infty(\iota)$ in $\mathfrak{H}_S$ is the same as $\mathfrak{H}_M$, and so $\mathfrak{H}_M \subset \mathfrak{H}_S$. If

$$\mathfrak{N}_S = \{f \in C^\nu(\iota) \cap \mathfrak{H}_S \mid Mf = 0\},$$

then

$$\mathfrak{H}_S = \mathfrak{H}_M \oplus \mathfrak{N}_S. \tag{3.14}$$

To show this suppose $f \in \mathfrak{H} \ominus \mathfrak{H}_M$. Then using (3.12) we have $(f, M\varphi)_2 = (f, \varphi) = 0$ for all $\varphi \in C_0^\infty(\iota)$ and the Corollary to Theorem 2.2 shows that $f \in \mathfrak{N}_S$. Reversing these steps shows that $\mathfrak{N}_S \subset \mathfrak{H}_S \ominus \mathfrak{H}_M$, and so $\mathfrak{H}_S \ominus \mathfrak{H}_M = \mathfrak{N}_S$. Note that $\mathfrak{N}_S$ is finite-dimensional with dim $\mathfrak{N}_S \leqslant \nu m$.

Let S_F be the Friedrichs extension of S, We claim that $\mathfrak{D}(S_F)$ is in $\mathfrak{H}_S$. For if $f \in \mathfrak{D}(S_F)$ there exists a sequence $f_n \in \mathfrak{D}(S)$ such that

$$\|f - f_n\|_2 \to 0, \qquad \|f_n - f_m\| = (M(f_n - f_m), f_n - f_m)_2 \to 0,$$

and so $\|f_n - f_m\|_{2,J} \to 0$. If F is the element of $\mathfrak{H}_S$ determined by f_n, then $\|F - f_n\|_{2,J} \to 0$ for each compact subinterval $J \subset \iota$ and $\|f - f_n\|_2 \to 0$ imply that $F = f$. Now for $f \in \mathfrak{D}(S_F)$ we have

$$\begin{aligned}\|f\|^2 = (f, f) &= \lim_{n,m}(f_n, f_m) = \lim_{n,m}(Mf_n, f_m)_2 \\ &= \lim_n(Mf_n, f)_2 = \lim_n(f_n, S_F f)_2 = (f, S_F f)_2 = (Mf, f)_2,\end{aligned}$$

and this shows that the inner product on $\mathfrak{D}(S_F) \subset \mathfrak{H}_S$ is

$$(f, g) = (Mf, g)_2, \qquad f, g \in \mathfrak{D}(S_F),$$

and also

$$(S_F f, f)_2 = (Mf, f)_2 \geqslant (c(J))^2 \| f \|_{2,J}^2, \qquad f \in \mathfrak{D}(S_F). \tag{3.15}$$

Therefore we see that $\mathfrak{H}_S = \mathfrak{H}_{S_F}$, that is, $\mathfrak{H}_S$ can also be viewed as the completion of $\mathfrak{D}(S_F)$ with the inner product $(S_F f, g)_2$, f, $g \in \mathfrak{D}(S_F)$. This shows that in considering Hilbert spaces of the type $\mathfrak{H}_S$ we may assume that $S = H$, a self-adjoint extension of M_0 in $L^2(\iota)$ which satisfies (3.10) and (3.11).

Now we can see that there exists an injection $G: L_0^2(\iota) \to \mathfrak{H}_S$ completely analogous to the injection $G_M: L_0^2(\iota) \to \mathfrak{H}_M$.

THEOREM 3.2. *There exists a linear map* $G: L_0^2(\iota) \to \mathfrak{H}_S$ *with the following properties*:

(i) $(f, h)_2 = (f, Gh)$, $f \in \mathfrak{H}_S$, $h \in L_0^2(\iota)$,

(ii) $GM\varphi = \varphi$, $\varphi \in C_0^\infty(\iota)$,

(iii) $MGh = h$, $h \in L_0^2(\iota)$,

(iv) G *is injective and* $\mathfrak{R}(G)$ *is dense in* $\mathfrak{H}_S$,

(v) $G_M = P_M G$, *where* P_M *is the orthogonal projection of* $\mathfrak{H}_S$ *onto* $\mathfrak{H}_M$.

Proof. The map $f \mapsto (f, h)_2$ with $h \in L_0^2(\iota)$ is a continuous linear functional on $\mathfrak{H}_S$. Now apply the same arguments as in the proof of Theorem 3.1. As to (v) we have

$$(f, P_M Gh) = (f, h)_2 = (f, G_M h), \qquad f \in \mathfrak{H}_M, h \in L_0^2(\iota),$$

which shows $P_M Gh = G_M h$, $h \in L_0^2(\iota)$.

We note that if $h \in C(\iota) \cap L_0^2(\iota)$ then $Gh \in C^\nu(\iota) \cap \mathfrak{H}_S$.

A simple example is given by $M = -D^2$, $m = 1$, with $\iota = (0, \infty)$. The maximal operator $M_{\max}$ for M in $L^2(\iota)$ has a domain

$$\mathfrak{D}(M_{\max}) = \{f \in L^2(\iota) \mid f' \in AC_{\mathrm{loc}}([0, \infty)), Mf \in L^2(\iota)\}.$$

Note that $f' \in L^2(\iota)$ for $f \in \mathfrak{D}(M_{\max})$. Since M is limit-point at ∞, the self-adjoint extensions of M_0 are obtained from $M_{\max}$ by imposing a homogeneous boundary condition at 0. They are all given as restrictions H_α of $M_{\max}$, where

$$\begin{aligned} \mathfrak{D}(H_\alpha) &= \{f \in \mathfrak{D}(M_{\max}) \mid f'(0) = \alpha f(0)\}, \qquad && \alpha \in \mathbb{R}, \\ &= \{f \in \mathfrak{D}(M_{\max}) \mid f(0) = 0\}, && \alpha = \infty. \end{aligned}$$

We have for $f, g \in \mathfrak{D}(H_\alpha)$

$$
\begin{aligned}
(H_\alpha f, g)_2 &= \alpha f(0)\bar{g}(0) + (f', g')_2, && \alpha \in \mathbb{R},\\
&= (f', g')_2, && \alpha = \infty.
\end{aligned}
$$

Clearly H_α is positive if $0 \leqslant \alpha \leqslant \infty$. If $\alpha < 0$ it is easy to show that $-\alpha^2$ is an eigenvalue for H_α, so that H_α is positive if and only if $\alpha \geqslant 0$. Thus for $\alpha \geqslant 0$ we see that $(H_\alpha f, g)_2$, $f, g \in \mathfrak{D}(H_\alpha)$, gives an inner product on $\mathfrak{D}(H_\alpha)$. Conditions (3.10), (3.11) can be verified for $S = H_\alpha$ when $0 < \alpha \leqslant \infty$. Instead of doing this, we shall identify concretely the completion of $\mathfrak{D}(H_\alpha)$ in case $0 < \alpha \leqslant \infty$.

Consider the set

$$
\begin{aligned}
\mathfrak{H}_\alpha &= \{f \in AC_{\text{loc}}([0, \infty)) \mid f' \in L^2(\iota)\}, && 0 < \alpha < \infty,\\
&= \{f \in AC_{\text{loc}}([0, \infty)) \mid f' \in L^2(\iota), f(0) = 0\}, && \alpha = \infty,
\end{aligned}
$$

with inner product

$$
\begin{aligned}
(f, g) &= \alpha f(0)\bar{g}(0) + (f', g')_2, && 0 < \alpha < \infty,\\
&= (f', g')_2, && \alpha = \infty.
\end{aligned}
$$

It is clear that (,) is a true inner product with a corresponding norm $\| \ \|$. We show that $\mathfrak{H}_\alpha$ is complete, and hence $\mathfrak{H}_\alpha$ is a Hilbert space. Consider the case $0 < \alpha < \infty$, and let $f_n \in \mathfrak{H}_\alpha$ be such that

$$
\| f_n - f_m \|^2 = \alpha \, | f_n(0) - f_m(0)|^2 + \int_0^\infty | f_n'(t) - f_m'(t)|^2 \, dt \to 0.
$$

Then there exist a constant A and a $g \in L^2(\iota)$ such that $f_n(0) \to A$ and $\| f_n' - g \|_2 \to 0$. If

$$
f(x) = A + \int_0^x g(t) \, dt,
$$

then $f \in \mathfrak{H}_\alpha$, and

$$
\| f - f_n \|^2 = \alpha \, | f(0) - f_n(0)|^2 + \int_0^\infty | g(t) - f_n'(t)|^2 \, dt \to 0,
$$

which demonstrates the completeness of $\mathfrak{H}_\alpha$ for $0 < \alpha < \infty$. Note that, since

$$
f(x) - f_n(x) = A - f_n(0) + \int_0^x (g(t) - f_n'(t)) \, dt \quad ,
$$

we have for every compact subinterval $J \subset [0, \infty)$

$$
| f(x) - f_n(x)| \leqslant | A - f_n(0)| + b(J) \| g - f_n' \|_2, \qquad x \in J,
$$

where $b(J) = \max\{x^{1/2} \mid x \in J\}$. Therefore $f_n \to f$ uniformly on compact subintervals of $[0, \infty)$. A similar argument applies to the space $\mathfrak{H}_\infty$.

For every compact subinterval $J \subset [0, \infty)$ there is a constant $c(J) > 0$ such that

$$\|f\| \geqslant c(J)\, \|f\|_{2,J}, \qquad f \in \mathfrak{H}_\alpha,\ 0 < \alpha \leqslant \infty.$$

To see this we note that for $f \in \mathfrak{H}_\alpha$ we have

$$f(x) = f(0) + \int_0^x f'(t)\, dt,$$

and this implies that

$$\begin{aligned} |f(x)|^2 &\leqslant \left(|f(0)| + \int_0^x |f'(t)|\, dt\right)^2 \\ &\leqslant 2\left[|f(0)|^2 + \left(\int_0^x |f'(t)|\, dt\right)^2\right] \\ &\leqslant 2[|f(0)|^2 + x\, \|f'\|_2^2], \end{aligned}$$

and hence for $0 < \alpha < \infty$

$$\begin{aligned} \int_J |f(x)|^2\, dx &\leqslant 2\left[|f(0)|^2\, |J| + \left(\int_J x\, dx\right)\left(\int_0^\infty |f'(t)|^2\, dt\right)\right] \\ &\leqslant d(J)[\alpha\, |f(0)|^2 + \|f'\|_2^2] \\ &= d(J)\, \|f\|^2, \end{aligned}$$

where $|J|$ is the length of J and

$$d(J) = 2 \max\left(|J|/\alpha, \int_J x\, dx\right).$$

In case $\alpha = \infty$ we have $f(0) = 0$ so that

$$\int_J |f(x)|^2\, dx \leqslant d(J)\, \|f\|^2,$$

where now $d(J) = 2 \int_J x\, dx$.

We observe that $\mathfrak{D}(H_\alpha) \subset \mathfrak{H}_\alpha$ if $0 < \alpha \leqslant \infty$, and that $(f, g) = (H_\alpha f, g)_2$, $f, g \in \mathfrak{D}(H_\alpha)$. We now show that $\mathfrak{D}(H_\alpha)$ is dense in $\mathfrak{H}_\alpha$. Let $f \in \mathfrak{H}_\alpha$ and $(f, \varphi) = 0$ for all $\varphi \in \mathfrak{D}(H_\alpha)$. In particular we have $0 = (f, \varphi) = (f', \varphi')_2$ for all $\varphi \in C_0^\infty(\iota)$. Thus $f(x) = a + bx$, and since $f \in \mathfrak{H}_\alpha$ we must have $b = 0$ for $0 < \alpha < \infty$ and $a = b = 0$ for $\alpha = \infty$. Hence the proof is complete in case $\alpha = \infty$, whereas for $0 < \alpha < \infty$ we must have $\alpha f(0)\, \bar{\varphi}(0) = 0$ for $\varphi \in \mathfrak{D}(H_\alpha)$, which implies $a = f(0) = 0$, completing the proof for $0 < \alpha < \infty$. Therefore $(\mathfrak{D}(H_\alpha))^c = \mathfrak{H}_\alpha$

for $0 < \alpha \leqslant \infty$, and $\mathfrak{H}_\infty$ is also the completion of $C_0^\infty(\iota)$. Thus $\mathfrak{H}_\infty = \mathfrak{H}_M$, and

$$\mathfrak{H}_\alpha = \mathfrak{H}_M \oplus \mathfrak{N}_\alpha, \qquad 0 < \alpha \leqslant \infty,$$

where $\mathfrak{N}_\alpha = \operatorname{span}\{1\}$ if $0 < \alpha < \infty$ and $\mathfrak{N}_\alpha = \{0\}$ if $\alpha = \infty$.

None of these $\mathfrak{H}_\alpha$, $0 < \alpha \leqslant \infty$, satisfies a global inequality of the form

$$\|f\| \geqslant c\|f\|_2, \qquad f \in \mathfrak{H}_\alpha, c > 0,$$

for if this were true, it would be true for $f \in C_0^\infty(\iota)$, and we have already shown that this is not possible. In case $\alpha = 0$ we have an inner product $(f, g) = (f', g')_2$ on $\mathfrak{D}(H_0)$, but the completion $\mathfrak{H}_0$ of $\mathfrak{D}(H_0)$ is not contained in $L^2_{\mathrm{loc}}(\iota)$. In fact, if $11/2 < \gamma < 6$, then

$$\begin{aligned} \varphi_n(x) &= n^{-\gamma}(n^2 - x^2)^3, \qquad && 0 \leqslant x \leqslant n, \\ &= 0, && x > n, \end{aligned}$$

gives a sequence $\varphi_n \in \mathfrak{D}(H_0)$ such that $\|\varphi_n\| \to 0$, but $\|\varphi_n\|_{2,J} \to \infty$ for each compact subinterval $J \subset [0, \infty)$.

This last example can be extended to show that assumption (3.2) does not hold for the case $M = -D^2$, $m = 1$, $\iota = \mathbb{R}$. Indeed, if $\varphi_n(-x) = \varphi_n(x)$ defines φ_n on $\mathbb{R}$, then a calculation shows that

$$\|\varphi_n\| = \|\varphi_n'\|_2 \to 0, \qquad \|\varphi_n\|_{2,J} \to \infty,$$

for each compact subinterval $J \subset \mathbb{R}$. If (3.2) were true, then $\varphi_n \in \mathfrak{H}_M$ by (3.6), and the above result contradicts (3.4).

A global inequality for $\mathfrak{H}_S$. It may happen that some symmetric extensions S of M_0 in $L^2(\iota)$ satisfy a stronger condition than (3.10), namely,

$$(Sf, f)_2 = (Mf, f)_2 \geqslant c^2(f, f)_2, \qquad f \in \mathfrak{D}(S), c > 0. \tag{3.16}$$

This implies, in particular, that

$$(Mf, f)_2 \geqslant c^2(f, f)_2, \qquad f \in C_0^\infty(\iota), c > 0.$$

Condition (3.16) was assumed by Brauer in his work on regular self-adjoint problems associated with pairs of differential operators [2]. This work was a continuation of earlier work by Kamke [17], who also considered the case $c = 0$. Krein, in [20], made an extensive study of ordinary differential operators in the regular case which are bounded below. In case (3.16) is valid the completion of $\mathfrak{D}(S)$ with the inner product $(f, g) = (Mf, g)_2$, $f, g \in \mathfrak{D}(S)$, is a Hilbert space $\mathfrak{H}_S$ and the identity map on $\mathfrak{D}(S)$ has an extension to $\mathfrak{H}_S$ into $L^2(\iota)$, which is injective. Moreover

$$(f, \varphi) = (f, M\varphi)_2, \qquad f \in \mathfrak{H}_S, \varphi \in \mathfrak{D}(S^c). \tag{3.17}$$

The proof follows that for the case $\mathfrak{H}_M$. If $f_n \in \mathfrak{D}(S)$ is a Cauchy sequence in $\mathfrak{H}_S$, there is an $F \in \mathfrak{H}_S$ and an $f \in L^2(\iota)$ such that $\|f_n - F\| \to 0$, $\|f_n - f\|_2 \to 0$. Then for $\varphi \in \mathfrak{D}(S)$

$$(F, \varphi) = \lim(f_n, \varphi) = \lim(f_n, M\varphi)_2 = (f, M\varphi)_2,$$

and hence $f = 0$ implies $(F, \varphi) = 0$ for all $\varphi \in \mathfrak{D}(S)$, a dense set in $\mathfrak{H}_S$, or $F = 0$. Thus the map $F \mapsto f$ is injective, and (3.17) is valid for $f \in \mathfrak{H}_S$, $\varphi \in \mathfrak{D}(S)$, which implies (3.17) itself.

Inequality (3.16) extends to $\mathfrak{H}_S$,

$$\|f\| \geqslant c\|f\|_2, \qquad f \in \mathfrak{H}_S.$$

On account of the inequality

$$|(f, h)_2| \leqslant \|f\|_2 \|h\|_2 \leqslant (c^{-1}\|h\|_2)\|f\|, \qquad f \in \mathfrak{H}_M,\ h \in L^2(\iota),$$

the mapping G has an extension, call it G also, to an injection $G: L^2(\iota) \to \mathfrak{H}_S$ such that Theorem 3.2 is valid with $L_0^2(\iota)$ replaced by $L^2(\iota)$ everywhere.

If $S^c = H$ is self-adjoint, then (3.16) for H implies that H^{-1} exists as a bounded operator on all of $L^2(\iota)$. Hence (3.17) and Theorem 3.2 (with $L_0^2(\iota)$ replaced by $L^2(\iota)$) gives

$$(H^{-1}h, \varphi) = (H^{-1}h, M\varphi)_2 = (h, \varphi)_2 = (Gh, \varphi), \qquad \varphi \in \mathfrak{D}(H),\ h \in L^2(\iota),$$

so that $G = H^{-1}$. In fact, in this case we have $\mathfrak{H}_H = \mathfrak{D}(H^{1/2})$.

A simple example occurs with $M = -D^2$, $m = 1$, $\iota = (0, 1)$. Then

$$\mathfrak{D}(M_{\max}) = \{f \in L^2(\iota) \mid f' \in AC(\bar{\iota}), f'' \in L^2(\iota)\}, \qquad \bar{\iota} = [0, 1].$$

Consider the self-adjoint extensions $H_{\alpha\beta}$ of M_0 in $L^2(\iota)$ given by separated boundary conditions

$$\begin{aligned}
\mathfrak{D}(H_{\alpha\beta}) &= \{f \in \mathfrak{D}(M_{\max}) \mid f'(0) = \alpha f(0), f'(1) = -\beta f(1)\}, && \alpha, \beta \in \mathbb{R},\\
&= \{f \in \mathfrak{D}(M_{\max}) \mid f'(0) = \alpha f(0), f(1) = 0\}, && \alpha \in \mathbb{R}, \beta = \infty,\\
&= \{f \in \mathfrak{D}(M_{\max}) \mid f(0) = 0, f'(1) = -\beta f(1)\}, && \alpha = \infty, \beta \in \mathbb{R},\\
&= \{f \in \mathfrak{D}(M_{\max}) \mid f(0) = 0, f(1) = 0\}, && \alpha = \infty, \beta = \infty.
\end{aligned}$$

The operators $H_{\alpha\beta}$ are special cases of those treated by Krein in [20]. We give a direct treatment of the basic facts about these simple operators.

The operator $H_{\alpha\beta}$ has zero as an eigenvalue if and only if

$$\alpha\beta + \alpha + \beta = 0, \qquad \alpha, \beta \in \mathbb{R} \cup \{\infty\},$$

and it has only positive eigenvalues if and only if $\{\alpha, \beta\} \in P$, where

$$\begin{aligned} P &= \{\{\alpha, \beta\} \in (\mathbb{R} \cup \{\infty\})^2 \mid \alpha\beta + \alpha + \beta > 0, \alpha + 1 > 0\} \\ &= \{\{\alpha, \beta\} \in (\mathbb{R} \cup \{\infty\})^2 \mid \alpha\beta + \alpha + \beta > 0, \beta + 1 > 0\}. \end{aligned}$$

Thus, for $\{\alpha, \beta\} \in P$, we have

$$(H_{\alpha\beta} f, f)_2 \geqslant \lambda_1 (f, f)_2, \qquad f \in \mathfrak{D}(H_{\alpha\beta}),$$

where $\lambda_1 > 0$ is the smallest eigenvalue of $H_{\alpha\beta}$. We have

$$\begin{aligned} (H_{\alpha\beta} f, g)_2 &= \alpha f(0)\bar{g}(0) + \beta f(1)\bar{g}(1) + (f', g')_2, && \alpha < \infty, \beta < \infty, \\ &= \alpha f(0)\bar{g}(0) + (f', g')_2, && \alpha < \infty, \beta = \infty, \\ &= \beta f(1)\bar{g}(1) + (f', g')_2, && \alpha = \infty, \beta < \infty, \\ &= (f', g')_2, && \alpha = \infty, \beta = \infty. \end{aligned}$$

For $\{\alpha, \beta\} \in P$ an inner product on $\mathfrak{D}(H_{\alpha\beta})$ is given by $(H_{\alpha\beta} f, g)_2$, $f, g \in \mathfrak{D}(H_{\alpha\beta})$.

In order to identify the completion of $\mathfrak{D}(H_{\alpha\beta})$ we first consider the set

$$\begin{aligned} \mathfrak{H}_{\alpha\beta} &= \{f \in AC(\bar{\iota}) \mid f' \in L^2(\iota)\} && \alpha < \infty, \beta < \infty, \\ &= \{f \in AC(\bar{\iota}) \mid f' \in L^2(\iota), f(1) = 0\}, && \alpha < \infty, \beta = \infty, \\ &= \{f \in AC(\bar{\iota}) \mid f' \in L^2(\iota), f(0) = 0\}, && \alpha = \infty, \beta < \infty, \\ &= \{f \in AC(\bar{\iota}) \mid f' \in L^2(\iota), f(0) = f(1) = 0\}, && \alpha = \infty, \beta = \infty, \end{aligned}$$

with inner product

$$\begin{aligned} (f, g) &= \alpha f(0)\bar{g}(0) + \beta f(1)\bar{g}(1) + (f', g')_2, && \alpha < \infty, \beta < \infty, \\ &= \alpha f(0)\bar{g}(0) + (f', g')_2, && \alpha < \infty, \beta = \infty, \\ &= \beta f(1)\bar{g}(1) + (f', g')_2, && \alpha = \infty, \beta < \infty, \\ &= (f', g')_2, && \alpha = \infty, \beta = \infty. \end{aligned}$$

Then $\mathfrak{H}_{\alpha\beta}$ is a Hilbert space for $\{\alpha, \beta\} \in P$. We show this for $\alpha < \infty, \beta < \infty$. If $f \in \mathfrak{H}_{\alpha\beta}$, then

$$\begin{aligned} \int_0^1 |f'|^2 &\geqslant |f(1) - f(0)|^2 \\ &\geqslant \max\left[\frac{\beta}{\beta+1} |f(0)|^2 - \beta |f(1)|^2, \frac{\alpha}{\alpha+1} |f(1)|^2 - \alpha |f(0)|^2\right] \end{aligned}$$

implies

$$\alpha |f(0)|^2 + \beta |f(1)|^2 + \|f'\|_2^2 \geqslant (\alpha\beta + \alpha + \beta) \max\left[\frac{|f(0)|^2}{\beta+1}, \frac{|f(1)|^2}{\alpha+1}\right],$$

so that $(\, , \,)$ is an inner product on $\mathfrak{H}_{\alpha\beta}$. If $\| \; \|$ is the corresponding norm, let $f_n \in \mathfrak{H}_{\alpha\beta}$ be such that

$$\| f_n - f_m \|^2 = \alpha \, | f_n(0) - f_m(0) |^2 + \beta \, | f_n(1) - f_m(1) |^2 + \| f'_n - f'_m \|_2^2 \to 0.$$

Then there exist constants A, B, and a $g \in L^2(\iota)$ such that $f_n(0) \to A$, $f_n(1) \to B$, $\| f'_n - g \|_2 \to 0$. If

$$f(x) = A + \int_0^x g(t) \, dt,$$

then we see that $f \in \mathfrak{H}_{\alpha\beta}$ and $f_n \to f$ uniformly on $\bar{\iota}$. In particular this shows that

$$\| f_n - f \|^2 = \alpha \, | f_n(0) - f(0) |^2 + \beta \, | f_n(1) - f(1) |^2 + \| f'_n - g \|_2^2 \to 0,$$

and hence $\mathfrak{H}_{\alpha\beta}$ is complete, and thus a Hilbert space. Similar arguments yield the result for other values of $\{\alpha, \beta\} \in P$. Now we note that $\mathfrak{D}(H_{\alpha\beta}) \subset \mathfrak{H}_{\alpha\beta}$ for $\{\alpha, \beta\} \in P$, and that $(f, g) = (H_{\alpha\beta} f, g)_2$, $f, g \in \mathfrak{D}(H_{\alpha\beta})$. In fact $\mathfrak{D}(H_{\alpha\beta})$ is dense in $\mathfrak{H}_{\alpha\beta}$; we indicate the reasoning in case $\alpha < \infty$, $\beta < \infty$. Let $f \in \mathfrak{H}_{\alpha\beta}$ satisfy $(f, \varphi) = 0$ for all $\varphi \in \mathfrak{D}(H_{\alpha\beta})$. Then $(f, \varphi) = 0$ for all $\varphi \in C_0^\infty(\iota)$, which implies $(f', \varphi')_2 = 0$ for all $C_0^\infty(\iota)$, so that $f(x) = a + bx$ for some $a, b \in \mathbb{C}$. (Note that $f \in \mathfrak{H}_{\alpha\beta}$ for all $a, b \in \mathbb{C}$.) Hence, we must have

$$(f, \varphi) = \alpha f(0) \, \bar{\varphi}(0) + \beta f(1) \, \bar{\varphi}(1) + (f', \varphi')_2 = 0, \qquad \varphi \in \mathfrak{D}(H_{\alpha\beta}),$$

or $\alpha a - b = 0$, $\beta a + (\beta + 1) b = 0$, which implies $a = b = 0$ or $f(x) = 0$, $x \in \bar{\iota}$. Using similar arguments for the remaining $\{\alpha, \beta\} \in P$ we obtain

$$\mathfrak{H}_{\alpha\beta} = \mathfrak{H}_M \oplus \mathfrak{N}_{\alpha\beta},$$

where

$$\begin{aligned} \mathfrak{N}_{\alpha\beta} &= \operatorname{span}\{1, x\}, && \alpha < \infty, \beta < \infty, \\ &= \operatorname{span}\{1 - x\}, && \alpha < \infty, \beta = \infty, \\ &= \operatorname{span}\{x\}, && \alpha = \infty, \beta < \infty, \\ &= \{0\}, && \alpha = \infty, \beta = \infty. \end{aligned}$$

Since $(\mathfrak{D}(H_{\alpha\beta}))^c = \mathfrak{H}_{\alpha\beta}$ we obtain the inequality

$$\| f \| \geqslant \lambda_1^{1/2} \| f \|_2, \qquad f \in \mathfrak{H}_{\alpha\beta},$$

where λ_1 is the smallest eigenvalue of $H_{\alpha\beta}$.

Remarks. 1. Let S be a densely defined, symmetric operator in a Hilbert space, bounded below with $m(S) \geqslant 0$, where

$$m(S) = \inf\{(Sf, f) \mid f \in \mathfrak{D}(S), \| f \| = 1\}.$$

The Friedrichs extension S_F of S is a self-adjoint operator with the property $m(S_F) = m(S)$. The spectrum of S_F is a closed subset of $\mathbb{R}$, and $m(S_F)$ is the lowest point of the spectrum of S_F (cf., e.g., [28, p. 190]).

2. Suppose S is a symmetric extension of M_0 in $L^2(\iota)$, such that $m(S) \geqslant 0$. Then S satisfies (3.16) if and only if $0 \in \rho(S_F)$, the resolvent set of S_F. In particular, if $\mathfrak{R}(M_{\min})$ is not closed, it is clear that no closed symmetric extension of $M_{\min}$ in $L^2(\iota)$ satisfies (3.16). As an example, consider $M = -D^2 + q$, $m = 1$, $\iota = (0, \infty)$, with q continuous on $[0, \infty)$ and $q \geqslant 0$. If $q \in L^1(0, \infty)$, then $\mathfrak{R}(M_{\min})$ is not closed (cf. [14, p. 1597]).

3. If S is a symmetric extension of M_0 in $L^2(\iota)$, such that (3.10) and (3.11) are valid, then $m(S) \geqslant 0$, and if we assume that $m(S_F) = m(S)$ is an eigenvalue of S_F, then it follows from (3.15) that $m(S_F) > 0$, which implies that S satisfies (3.16). In particular, if S is a symmetric extension of M_0 in $L^2(\iota)$, $m(S) \geqslant 0$, such that S_F has pure point spectrum, then (3.10) and (3.11) imply (3.16). In the regular case, where $\bar{\iota}$ is a compact interval, $Q_k(m \times m) \in C^k(\bar{\iota})$, $k = 0,\ldots, \nu$, and $Q_\nu(x)$ is invertible for $x \in \bar{\iota}$, every self-adjoint extension of M_0 in $L^2(\iota)$ has pure point spectrum. Hence in that case it is no restriction to assume (3.16) instead of (3.10) and (3.11).

Dirichlet integrals. We have seen that on $C_0^\infty(\iota)$ the inner product $(Mf, g)_2$, $f, g \in C_0^\infty(\iota)$ is a Dirichlet integral. There may be larger sets than $C_0^\infty(\iota)$ on which the Dirichlet integral provides an inner product. This situation has been considered by Pleijel and Bennewitz (cf., e.g., [23, 1]). Here we will consider a case, where such a "Dirichlet inner product" is produced by a self-adjoint extension of M_0 in $L^2(\iota)$, which satisfies (3.10) and (3.11).

We consider $M = -D^2 + q$, $m = 1$, $\iota = (0, \infty)$, with q continuous on $[0, \infty)$ such that

$$q(x) > 0, \qquad x \in \iota.$$

This implies that M is limit-point at ∞ (cf., e.g., [12, p. 231]). Then we have

$$(Mf, g)_2 = f'(0)\,\bar{g}(0) + \int_0^\infty (f'\bar{g}' + qf\bar{g}), \qquad f, g \in \mathfrak{D}(M_{\max}),$$

for $f'(x)\bar{g}(x) \to 0$ as $x \to \infty$ [15]. We consider the self-adjoint extension H_0 of M_0 given by

$$\mathfrak{D}(H_0) = \{f \in \mathfrak{D}(M_{\max}) \mid f'(0) = 0\}.$$

Note that

$$(H_0 f, g)_2 = \int_0^\infty (f'\bar{g}' + qf\bar{g}), \qquad f, g \in \mathfrak{D}(H_0).$$

From this it is clear that H_0 satisfies (3.10). We verify (3.11) by identifying the completion of $\mathfrak{D}(H_0)$ with the inner product $(H_0 f, g)_2$, $f, g \in \mathfrak{D}(H_0)$, directly. Therefore we introduce the set

$$\mathfrak{H}_0 = \{f \in AC_{\text{loc}}([0, \infty)) \mid f' \in L^2(\iota), q^{1/2} f \in L^2(\iota)\}$$

with inner product

$$(f, g) = \int_0^\infty (f'\bar{g}' + q f \bar{g}), \qquad f, g \in \mathfrak{H}_0 .$$

It is clear that (,) is a true inner product and that $\mathfrak{D}(H_0) \subset \mathfrak{H}_0$ while $(f, g) = (H_0 f, g)_2$ for $f, g \in \mathfrak{D}(H_0)$. We show that $\mathfrak{H}_0$ is complete and hence is a Hilbert space. Let $f_n \in \mathfrak{H}_0$ be such that

$$\| f_n - f_m \|^2 = \| f_n' - f_m' \|_2^2 + \| q^{1/2}(f_n - f_m)\|_2^2 \to 0.$$

Then there exist functions f and g such that $q^{1/2} f \in L^2(\iota)$, $q^{1/2} f_n \to q^{1/2} f$ in $L^2(\iota)$, and $g \in L^2(\iota)$, $f_n' \to g$ in $L^2(\iota)$. For $\varphi \in C_0^\infty(\iota)$ having support in J we have

$$\begin{aligned} \int_\iota f\bar{\varphi}' &= \int_J f\bar{\varphi}' = \lim \int_J f_n \bar{\varphi}' = -\lim \int_J f_n' \bar{\varphi} \\ &= -\int_J g\bar{\varphi} = -\int_\iota g\bar{\varphi}, \end{aligned}$$

so that

$$(f, \varphi')_2 = -(g, \varphi)_2 , \qquad \varphi \in C_0^\infty(\iota).$$

Hence $f \in AC_{\text{loc}}([0, \infty))$ and $g = f'$, which shows $f \in \mathfrak{H}_0$. From

$$\| f - f_n \|^2 = \| f' - f_n' \|_2^2 + \| q^{1/2}(f - f_n)\|_2^2 \to 0, \qquad n \to \infty,$$

it follows that $\mathfrak{H}_0$ is a Hilbert space.

We now show that $\mathfrak{D}(H_0)$ is dense in $\mathfrak{H}_0$. Let $f \in \mathfrak{H}_0$ and $(f, \varphi) = 0$ for all $\varphi \in \mathfrak{D}(H_0)$. In particular we have $(f, \varphi) = (f, M\varphi)_2 = 0$ for all $\varphi \in C_0^\infty(\iota)$. This shows $f'' = qf$. Now let $\varphi \in \mathfrak{D}(H_0)$ be such that $\varphi(x) = 0$ for all $x > p$ for some $p > 0$. Then

$$(f, \varphi) = -f'(0)\,\bar{\varphi}(0) + \int_0^\infty (-f'' + qf)\bar{\varphi} = -f'(0)\,\bar{\varphi}(0),$$

which shows $f'(0) = 0$. In order to demonstrate the fact that $f = 0$ we can assume that f is real valued. Since $f \in \mathfrak{H}_0$ it is clear that $\lambda = \lim_{x\to\infty} f(x) f'(x)$ exists as a real number and that we have

$$0 = (Mf, f)_2 = -\lambda + \int_0^\infty (| f' |^2 + q\, | f |^2).$$

If $\lambda \leqslant 0$, then it follows that $f = 0$. Next we show that $\lambda > 0$ is not possible. If $\lambda > 0$ there exists a number c such that, for all $x > c$, $f(x)f'(x) > \lambda/2$ and

$$f^2(x) - f^2(0) = \int_0^x (f^2)' = 2\int_0^x ff' = 2\int_0^c ff' + 2\int_c^x ff'$$
$$\geqslant 2\int_0^c ff' + \lambda(x - c), \qquad x > c.$$

Hence for some $a \in \mathbb{R}$

$$f^2(x) \geqslant \lambda(x + a), \qquad x > c.$$

First we consider the case

$$\int_1^\infty xq(x)\,dx = \infty.$$

Then

$$\int_c^\infty q(x)f^2(x)\,dx \geqslant \lambda\int_c^\infty (x + a)\,q(x)\,dx = \infty,$$

which leads to a contradiction, since we must have $q^{1/2}f \in L^2(\iota)$. Next we consider the case

$$\int_1^\infty xq(x)\,dx < \infty.$$

In this case the equation $Mu = 0$ has two solutions, u_1 and u_2, such that

$$\lim_{x\to\infty} u_1(x) = 1, \qquad \lim_{x\to\infty} u_1'(x) = 0,$$

and

$$\lim_{x\to\infty} u_2(x)/x = 1, \qquad \lim_{x\to\infty} u_2'(x) = 1,$$

(cf. [12, p. 103, exercise 28]). Hence there exist real numbers a and b such that $f = au_1 + bu_2$. From $f' = au_1' + bu_2'$, it follows that $\lim_{x\to\infty} f'(x) = b$; in order that $f \in \mathfrak{H}_0$ we must have $f' \in L^2(\iota)$ so that $b = 0$. But then

$$f(x)f'(x) = a^2u_1(x)\,u_1'(x) \to 0$$

as $x \to \infty$. This implies that $\lambda > 0$ is not possible, and therefore $\mathfrak{D}(H_0)$ is dense in $\mathfrak{H}_0$.

If we let

$$\mathfrak{H}_\infty = \{f \in \mathfrak{H}_0 \mid f(0) = 0\},$$

then $\mathfrak{H}_\infty \subset \mathfrak{H}_0$ and with the inner product $(\ ,\ )$ $\mathfrak{H}_\infty$ is a closed subspace of $\mathfrak{H}_0$.

In order to see this let $f_n \in \mathfrak{H}_\infty$ be such that $f_n \to f$ in $\mathfrak{H}_0$. Then $\| f_n - f_m \| \to 0$, and since

$$f_n(x) = \int_0^x f'_n(t)\, dt$$

it follows that for every compact subinterval $J \subset [0, \infty)$

$$| f_n(x) - f_m(x) | \leqslant b(J) \| f'_n - f'_m \|_2 , \qquad x \in J,$$

where $b(J) = \max\{x^{1/2} \mid x \in J\}$. Therefore $f_n \to f$ uniformly on compact subintervals of $[0, \infty)$. In particular we have $f(0) = 0$, so that $f \in \mathfrak{H}_\infty$. Let H_∞ be the self-adjoint extension of M_0 in $L^2(\iota)$ given by

$$\mathfrak{D}(H_\infty) = \{f \in \mathfrak{D}(M_{\max}) \mid f(0) = 0\}.$$

It is clear that $C_0^\infty(\iota) \subset \mathfrak{D}(H_\infty) \subset \mathfrak{H}_\infty$. We show that $C_0^\infty(\iota)$ is dense in $\mathfrak{H}_\infty$. Let $f \in \mathfrak{H}_\infty$ be such that $(f, \varphi) = 0$ for all $\varphi \in C_0^\infty(\iota)$. Then $Mf = 0$, $f(0) = 0$, and the reasoning used to show $(\mathfrak{D}(H_0))^c = \mathfrak{H}_0$ can be applied to show $f = 0$. Thus $\mathfrak{H}_M = \mathfrak{H}_\infty$ is the completion of $\mathfrak{D}(H_\infty)$ with $(,)$ as an inner product. We have $\mathfrak{H}_0 = \mathfrak{H}_M \oplus \mathfrak{N}_0$, where

$$\mathfrak{N}_0 = \{f \in \mathfrak{H}_0 \mid Mf = 0\},$$

and it is clear that $\dim \mathfrak{N}_0 = \dim(\mathfrak{H}_0/\mathfrak{H}_M) = 1$.

We note that H_∞ is the Friedrichs extension M_F of M_0. To show this we observe that $\mathfrak{D}(M_F) \subset \mathfrak{H}_M = \mathfrak{H}_\infty$ (see (3.6)). Thus $f \in \mathfrak{D}(M_F)$ implies $f(0) = 0$, and hence $f \in \mathfrak{D}(H_\infty)$. From this it follows that $M_F \subset H_\infty$, and since both of these operators are self-adjoint we must have $M_F = H_\infty$.

4. Subspaces Determined by Pairs of Ordinary Differential Expressions

Let

$$M = \sum_{k=0}^{\nu} Q_k D^k$$

on $\iota = (a, b)$ be a formally symmetric ($M = M^+$) differential expression of even order $\nu = 2\mu$, where $Q_k(m \times m) \in C^k(\iota)$, $k = 0,..., \nu$, and $Q_\nu(x)$ is invertible for $x \in \iota$. Let H be a self-adjoint extension of M_0 in $L^2(\iota)$ such that (3.10), (3.11) are valid with $S = H$, so that (3.2) holds, and let $\mathfrak{H} = \mathfrak{H}_H$. Thus we have the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_M \oplus \mathfrak{N}_H$, and the injection G of $L_0^2(\iota)$ into $\mathfrak{H}$ satisfies the properties listed in Theorem 3.2.

On ι we consider another, not necessarily formally symmetric, differential expression L of order n,

$$L = \sum_{k=0}^{n} P_k D^k,$$

where $P_k(m \times m) \in C^k(\iota)$, $k = 0,..., n$, and where we assume

$$P_n(x) \textit{ invertible}, \qquad x \in \iota, \qquad \textit{if } n > \nu. \tag{4.1}$$

The pairs L, M and L^+, M naturally determine certain maximal and minimal linear manifolds in the Hilbert space $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$. The *maximal linear manifolds* are defined by

$$\begin{aligned} T &= \{\{f, g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), g \in C^\nu(\iota), Lf = Mg\}, \\ T^+ &= \{\{f, g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), g \in C^\nu(\iota), L^+f = Mg\}, \end{aligned} \tag{4.2}$$

where $r = \max(n, \nu)$, and where we understand that for $r \geqslant 1$, $\nu = 0$ these manifolds are given by

$$\begin{aligned} T &= \{\{f, g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), Lf = Q_0 g\}, \\ T^+ &= \{\{f, g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), L^+f = Q_0 g\}, \end{aligned}$$

and, in case $r = 0$, by

$$\begin{aligned} T &= \{\{f, g\} \in \mathfrak{H}^2 \mid P_0 f = Q_0 g\}, \\ T^+ &= \{\{f, g\} \in \mathfrak{H}^2 \mid P_0^* f = Q_0 g\}. \end{aligned}$$

It is clear that $T(0) = T^+(0) = \mathfrak{N}_H$, and hence T and T^+ are operators if and only if $\mathfrak{N}_H = \{0\}$. The *minimal linear manifolds* are defined by

$$\begin{aligned} S &= \{\{\varphi, \psi\} \in \mathfrak{H}^2 \mid \varphi \in C_0^\infty(\iota), \psi = GL\varphi\}, \\ S^+ &= \{\{\varphi, \psi\} \in \mathfrak{H}^2 \mid \varphi \in C_0^\infty(\iota), \psi = GL^+\varphi\}. \end{aligned} \tag{4.3}$$

Hence S and S^+ are operators with domain $C_0^\infty(\iota)$, and therefore S and S^+ are densely defined if and only if $\mathfrak{N}_H = \{0\}$.

THEOREM 4.1. *The minimal linear manifolds can also be described as follows:*

(i) $S = \{\{\varphi, \psi\} \in T \mid \varphi \in C_0^\infty(\iota), (\psi, f) = (M\psi, f)_2 \textit{ for all } f \in \mathfrak{N}_H\}$,

(ii) $S^+ = \{\{\varphi, \psi\} \in T^+ \varphi \in C_0^\infty(\iota), (\psi, f) = (M\psi, f)_2 \textit{ for all } f \in \mathfrak{N}_H\}$.

Proof. We will only prove (i), since the proof of (ii) is similar. If $\{\varphi, \psi\} \in S$ then $\psi = GL\varphi$, which implies $\psi \in C^\nu(\iota)$ and $L\varphi = M\psi$, so that $\{\varphi, \psi\} \in T$. Now for all $f \in \mathfrak{H}$, and in particular for all $f \in \mathfrak{N}_H$,

$$(\psi, f) - (M\psi, f)_2 = (GL\varphi, f) - (L\varphi, f)_2 = (L\varphi, f)_2 - (L\varphi, f)_2 = 0,$$

since $\varphi \in C_0^\infty(\iota)$ and $L\varphi \in L_0^2(\iota)$. Conversely, let $\{\varphi, \psi\} \in T$, $\varphi \in C_0^\infty(\iota)$, and $(\psi, f) = (M\psi, f)_2$ for all $f \in \mathfrak{N}_H$. If $u = \psi - GL\varphi$, then $u \in C^\nu(\iota) \cap \mathfrak{H}$ and

$$(u, f) = (\psi, f) - (GL\varphi, f) = (M\psi, f)_2 - (L\varphi, f)_2 = 0$$

for all $f \in \mathfrak{N}_H$. Hence $u \in \mathfrak{H}_M$. However $Mu = M\psi - L\varphi = 0$, and therefore $u = 0$. Thus $\psi = GL\varphi$, and $\{\varphi, \psi\} \in S$.

We note that the above proof actually shows that $\mathfrak{N}_H$ may be replaced by $\mathfrak{H}$ in (i) and (ii).

We now investigate the adjoints S^* and $(S^+)^*$ of S, S^+, respectively. For this, note that

$$(g, \varphi) - (f, \psi) = (g, M\varphi)_2 - (f, L\varphi)_2, \qquad \{f, g\} \in \mathfrak{H}^2, \{\varphi, \psi\} \in S, \tag{4.4}$$

and

$$(g, \varphi) - (f, \psi) = (g, M\varphi)_2 - (f, L^+\varphi)_2, \qquad \{f, g\} \in \mathfrak{H}^2, \{\varphi, \psi\} \in S^+. \tag{4.5}$$

As to (4.4), since $\varphi \in C_0^\infty(\iota)$ we have $L\varphi \in L_0^2(\iota)$, and then (3.12) and Theorem 3.2(i) imply that

$$(g, \varphi) - (f, \psi) = (g, M\varphi)_2 - (f, GL\varphi) = (g, M\varphi)_2 - (f, L\varphi)_2 .$$

Formula (4.5) follows similarly. A consequence of these equalities is that

$$\begin{aligned} S^* &= \{\{f, g\} \in \mathfrak{H}^2 \mid (g, M\varphi)_2 = (f, L\varphi)_2 \text{, all } \varphi \in C_0^\infty(\iota)\}, \\ (S^+)^* &= \{\{f, g\} \in \mathfrak{H}^2 \mid (g, M\varphi)_2 = (f, L^+\varphi)_2 \text{, all } \varphi \in C_0^\infty(\iota)\}, \end{aligned} \tag{4.6}$$

and this shows that $S^*((S^+)^*)$ is the set of all pairs $\{f, g\} \in \mathfrak{H}^2$ satisfying $L^+f = Mg(Lf = Mg)$ in the above weak sense. Actually, $S^* = (T^-)^c$ and $(S^+)^* = T^c$, as we shall prove below. First we remark that

$$S \subset T \subset T^c \subset (S^+)^*, \qquad S^+ \subset T^+ \subset (T^+)^c \subset S^*. \tag{4.7}$$

We have seen in Theorem 4.1 that $S \subset T \subset T^c$. Since $(S^+)^*$ is closed, it is sufficient to show $T \subset (S^+)^*$. If $\{f, g\} \in T$, then a partial integration of the terms on the right side of (4.5) yields

$$(g, \varphi) - (f, \psi) = (Mg, \varphi)_2 - (Lf, \varphi)_2 = 0 \qquad \text{for all } \{\varphi, \psi\} \in S^+,$$

and hence $\{f, g\} \in (S^+)^*$. The second set of inclusions in (4.7) follow from (4.4).

THEOREM 4.2. *We have*

(i) (a) $S^* \ominus S^+ = T^+ \cap JT$

$= \{\{f, g\} \in \mathfrak{H}^2 \mid f, g \in C^r(\iota), L^+f = Mg, Mf = -Lg\}$,

(b) $\mathfrak{D}(S^* \ominus S^+) = \nu(TT^+ + I)$, $\mathfrak{R}(S^* \ominus S^+) = \nu(T^+T + I)$,

(c) $S^* = (T^+)^c$,

(ii) (a) $(S^+)^* \ominus S = T \cap JT^+$
$$= \{\{f, g\} \in \mathfrak{H}^2 \mid f, g \in C^r(\iota), Lf = Mg, Mf = -L^+g\},$$
(b) $\mathfrak{D}((S^+)^* \ominus S) = \nu(T^+T + I), \mathfrak{R}((S^+)^* \ominus S) = \nu(TT^+ + I)$,
(c) $(S^+)^* = T^c$.

Proof. We prove (i), since (ii) is similar. First we note that $\{f, g\} \in S^* \ominus S^+$ if and only if $\{f, g\} \in \mathfrak{H}^2$ and

$$-(f, L\varphi)_2 + (g, M\varphi)_2 = 0, \quad (f, M\chi)_2 + (g, L^+\chi)_2 = 0, \quad \varphi, \chi \in C_0^\infty(\iota). \tag{4.8}$$

To see this, let $\{f, g\} \in S^* \ominus S^+ = S^* \cap (S^+)^\perp \subset \mathfrak{H}^2$. The first equality follows from the fact that $\{f, g\} \in S^*$ and (4.6), and the second results from the fact that $\{f, g\} \in (S^+)^\perp$,

$$0 = (f, \chi) + (g, GL^+\chi) = (f, M\chi)_2 + (g, L^+\chi)_2, \qquad \chi \in C_0^\infty(\iota),$$

where we are using (3.12) and Theorem 3.2(i). This shows that $S^* \ominus S^+$ is contained in the set of $\{f, g\} \in \mathfrak{H}^2$ satisfying (4.8), and the reverse inclusion follows by retracing the steps.

Now let us consider a differential expression $\mathscr{L}$, acting on $2m \times 1$ matrix-valued functions, given by

$$\mathscr{L} = \begin{pmatrix} -L & M \\ M & L^+ \end{pmatrix} = \sum_{k=0}^{r} \mathscr{P}_k D^k.$$

If

$$F = \begin{pmatrix} f \\ g \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

represent functions on $\iota \to \mathbb{C}^{2m}$, we see that (4.8) just says that

$$(F, \mathscr{L}\Phi)_2 = 0, \qquad \Phi \in C_0^\infty(\iota).$$

Clearly the coefficients $\mathscr{P}_k \in C^k(\iota)$, $k = 0,..., r$. The leading coefficient of $\mathscr{L}$ is the $2m \times 2m$ matrix-valued function $\mathscr{P}_r$ given by

$$\begin{aligned} \mathscr{P}_r &= \begin{pmatrix} -P_n & O_m{}^m \\ O_m{}^m & (-1)^n P_n^* \end{pmatrix}, \qquad r = n > \nu, \\ &= \begin{pmatrix} O_m{}^m & Q_\nu \\ Q_\nu & O_m{}^m \end{pmatrix}, \qquad r = \nu > n, \\ &= \begin{pmatrix} -P_\nu & Q_\nu \\ Q_\nu & P_\nu^* \end{pmatrix}, \qquad r = n = \nu = 2\mu. \end{aligned}$$

We claim that $\mathscr{P}_r(x)$ is invertible for all $x \in \iota$. If $n > \nu$ this is clear since P_n is invertible in this case, and if $n < \nu$ the invertibility of Q_ν implies that of $\mathscr{P}_r$.

Suppose $n = \nu$ and

$$\mathscr{P}_r \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -P_\nu \alpha + Q_\nu \beta \\ Q_\nu \alpha + P_\nu^* \beta \end{pmatrix} = 0_{2m}^1,$$

for some $\alpha, \beta \in \mathbb{C}^m$. Since Q_ν is invertible we must have $\beta = Q_\nu^{-1} P_\nu \alpha$ and then

$$Q_\nu \alpha + P_\nu^* Q_\nu^{-1} P_\nu \alpha = O_m^1,$$

which implies

$$\alpha^* R_\nu \alpha + \alpha^* P_\nu^* R_\nu^{-1} P_\nu \alpha = \alpha^* R_\nu \alpha + (P_\nu \alpha)^* R_\nu^{-1} (P_\nu \alpha) = 0, \tag{4.9}$$

where $R_\nu = (-1)^\mu Q_\nu$. But $R_\nu > 0$, and hence $R_\nu^{-1} > 0$, which shows that each of the terms in (4.9) must be 0, or $\alpha^* R_\nu \alpha = 0$. This implies $\alpha = 0$, and then $\beta = 0$. Hence $\mathscr{P}_r$ is invertible on ι in case $n = \nu$. Now we can apply the Corollary to Theorem 2.2 and obtain $F \in C^r(\iota)$ and $\mathscr{L}^+ F = 0_{2m}^1$. Thus $f, g \in C^r(\iota)$, and since

$$\mathscr{L}^+ = \begin{pmatrix} -L^+ & M \\ M & L \end{pmatrix},$$

we have $L^+ f = Mg$, $Mf = -Lg$. This implies $\{f, g\} \in T^+$ and $\{g, -f\} \in T$, or $\{f, g\} \in T^+ \cap JT$. Conversely, if $\{f, g\} \in T^+ \cap JT$, then $\{f, g\} \in S^*$ by (4.7), and $\{f, g\} \in JT$ implies

$$(f, \chi) + (g, GL^+\chi) = (f, M\chi)_2 + (g, L^+\chi)_2 = (Mf, \chi)_2 + (Lg, \chi)_2 = 0$$

for all $\chi \in C_0^\infty(\iota)$, so that $\{f, g\} \in (S^+)^\perp$. Thus $\{f, g\} \in S^* \ominus S^+$, and this completes the proof of (ia).

The first equality in (ib) follows from (ia), since $f \in \mathfrak{D}(S^* \ominus S^+)$ if and only if $\{f, g\} \in S^* \ominus S^+ = T^+ \cap JT$ for some $g \in \mathfrak{H}$, and this is true if and only if $\{f, g\} \in T^+, \{g, -f\} \in T$, or $\{f, -f\} \in TT^+$, or $f \in \nu(TT^+ + I)$. Since $J(S^* \ominus S^+) = (S^+)^* \ominus S$, we have that the second equality follows from the first one in (iib).

As to $S^* = (T^+)^c$ we already know by (4.7) that $(T^+)^c \subset S^*$. We have $S^* = (S^+)^c \oplus (S^* \ominus S^+)$, and $S^+ \subset T^+$, $S^* \ominus S^+ \subset T^+$ imply $S^* \subset (T^+)^c$, and this yields $S^* = (T^+)^c$.

We now define the *minimal subspaces* T_0, T_0^+ by $T_0 = S^c$, $T_0^+ = (S^+)^c$, and the *maximal subspaces* T_1, T_1^+ by $T_1 = T^c$, $T_1^+ = (T^+)^c$. Then we have

$$T_0 \subset T_1 = (T_0^+)^*, \qquad T_0^+ \subset T_1^+ = T_0^*,$$

and Theorem 4.2 shows that we can view T and T^+ as smooth versions of $(T_0^+)^*$ and T_0^*, respectively. Since we know $T_1 = T_0 \oplus (T \cap JT^+)$, $T_1^+ = T_0^+ \oplus (T^+ \cap JT)$, the nonsmooth elements in T_1, T_1^+ arise only from those $\{f, g\} \in T_0 \backslash S$ and $\{f, g\} \in T_0^+ \backslash S^+$.

Since

$$T_0(0) = (\mathfrak{D}(T_0^*))^\perp = (\mathfrak{D}((T^+)^c))^\perp = (\mathfrak{D}(T^+))^\perp,$$

and similarly $T_0{}^+(0) = (\mathfrak{D}(T))^\perp$, we see that T_0, $T_0{}^+$ are operators if and only if $\mathfrak{D}(T^+)$, $\mathfrak{D}(T)$ are dense in $\mathfrak{H}$, respectively. We do not know of an example such that T_0 or $T_0{}^+$ is not an operator.

Now we wish to show that $T_1(0)$, $T_1{}^+(0)$, $\nu(T_1 - lI)$, $\nu(T_1{}^+ - \bar{l}I)$ consist of smooth elements for most $l \in \mathbb{C}$. The set of all $l \in \mathbb{C}$ for which the leading coefficient of $L - lM$ is not invertible for some $x \in \iota$ will be denoted by $\mathbb{C}_e$, and called the *exceptional set* in $\mathbb{C}$. For $n > \nu$ the leading coefficient of $L - lM$ is P_n, for $n < \nu$ it is $-lQ_\nu$, and for $n = \nu$ it is $P_\nu - lQ_\nu$. Thus

$$\begin{aligned} \mathbb{C}_e &= \varnothing, && n > \nu, \\ &= \{0\}, && n < \nu, \\ &= \bigcup_{x \in \iota} \sigma(Q_\nu^{-1}(x)\, P_\nu(x)), && n = \nu, \end{aligned}$$

where $\sigma(A)$ denotes the spectrum of a matrix A, that is, the set of its eigenvalues. For $l \in \mathbb{C}\backslash\mathbb{C}_e$ the leading coefficients of $L - lM$ and of $L^+ - \bar{l}M$ are invertible for all $x \in \iota$.

THEOREM 4.3. *We have*

(i) $T_1(0) = T_1{}^+(0) = T(0) = T^+(0) = \mathfrak{N}_H = \{f \in C^\nu(\iota) \cap \mathfrak{H} \mid Mf = 0\}$,

(ii) *for* $l \in \mathbb{C}\backslash\mathbb{C}_e$,

$$\nu(T_1 - lI) = \nu(T - lI) = \{f \in C^r(\iota) \cap \mathfrak{H} \mid (L - lM)f = 0\},$$
$$\nu(T_1{}^+ - \bar{l}I) = \nu(T^+ - \bar{l}I) = \{f \in C^r(\iota) \cap \mathfrak{H} \mid (L^+ - \bar{l}M)f = 0\},$$

(iii) *and for* $l \in \mathbb{C}$,

$$T - lI = \{\{f, g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), g \in C^\nu(\iota), (L - lM)f = Mg\},$$
$$T^+ - \bar{l}I = \{\{f, g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), g \in C^\nu(\iota), (L^+ - \bar{l}M)f = Mg\}.$$

Proof. We know that $T(0) = T^+(0) = \mathfrak{N}_H$ and that $T(0) \subset T_1(0)$, $T^+(0) \subset T_1{}^+(0)$. Therefore we just have to show that $T_1(0) \subset \mathfrak{N}_H$ and $T_1{}^+(0) \subset \mathfrak{N}_H$. If $g \in T_1(0)$, or $\{0, g\} \in T_1$, then (4.6) shows that $(g, M\varphi)_2 = 0$ for all $\varphi \in C_0{}^\infty(\iota)$, and the corollary to Theorem 2.2 implies that $g \in C^\nu(\iota)$, $Mg = 0$, or $g \in \mathfrak{N}_H$. Thus $T_1(0) = \mathfrak{N}_H$, and $T_1{}^+(0) = \mathfrak{N}_H$ follows similarly, proving (i). For (ii) we have $\nu(T - lI) \subset \nu(T_1 - lI)$, and so we just have to show the reverse inclusion. Suppose $\{f, 0\} \in T_1 - lI$, or $\{f, lf\} \in T_1$. Then (4.6) implies that $(f, (L^+ - \bar{l}M)\varphi)_2 = 0$ for all $\varphi \in C_0{}^\infty(\iota)$. For $l \in \mathbb{C}\backslash\mathbb{C}_e$ we can apply the corollary to Theorem 2.2 to obtain $f \in C^r(\iota)$ and $(L - lM)f = 0$, which shows that $\{f, lf\} \in T$ and

$f \in \nu(T - lI)$. The proof of the second set of equalities in (ii) is similar. As to (iii) we have $\{f, g\} \in T - lI$ if and only if $\{f, g + lf\} \in T$. Thus $f \in C^r(\iota)$ and $g + lf \in C^\nu(\iota)$, which implies that $g \in C^\nu(\iota)$ and $Lf = M(g + lf)$, or $(L - lM)f = Mg$. This argument may be reversed, and also used for T^+. Thus (iii) is demonstrated, and the proof of the theorem is complete.

Remarks. From (i) we have that T_1, T_1^+, and hence T_0, T_0^+, are operators if $\mathfrak{N}_H = \{0\}$. In particular if $\nu = 0$ and $Mf = Q_0 f$, where $Q_0(x) > 0$ for $x \in \iota$, then $\mathfrak{N}_H = \{0\}$. In the degenerate case $r = 0$ we can see that

$$T = \{\{f, Q_0^{-1}P_0 f\} \in \mathfrak{H}^2, \qquad T^+ = \{\{f, Q_0^{-1}P_0^* f\} \in \mathfrak{H}^2\},$$

$$S = \{\{\varphi, Q_0^{-1}P_0\varphi\} \mid \varphi \in C_0^\infty(\iota)\}, \qquad S^+ = \{\{\varphi, Q_0^{-1}P_0^*\varphi\} \mid \varphi \in C_0^\infty(\iota)\}.$$

In this case $T_1 = T_0$ and $T_1^+ = T_0^+$, for from Theorem 4.2 we have that $T_1^+ \ominus T_0^+$ is the set of all those $\{f, g\} \in \mathfrak{H}^2$ satisfying

$$\mathscr{P}\begin{pmatrix} f \\ g \end{pmatrix} = O_{2m}^1, \qquad \mathscr{P}_0 = \begin{pmatrix} -P_0 & Q_0 \\ Q_0 & P_0^* \end{pmatrix},$$

and the invertibility of $\mathscr{P}_0$ on ι implies $f = g = 0$, or $T_1^+ = T_0^+$. Similarly, $T_1 \ominus T_0$ is trivial.

We are now in a position to study subspace extensions A of T_0. These can be in $\mathfrak{H}^2$ itself, or in a larger space $\mathfrak{K}^2 \supset \mathfrak{H}^2$, where $\mathfrak{K}$ is a Hilbert space containing $\mathfrak{H} = \mathfrak{H}_H$. Let us look at the situation where the subspace A satisfies $T_0 \subset A \subset T_1$. If we put

$$t = \dim(T_1/T_0) = \dim(T_1^+/T_0^+) = \dim(T \cap JT^+) = \dim(T^+ \cap JT),$$

then clearly $t \leqslant 2rm$. We can now abstractly characterize all adjoint pairs A, A^*. This result is the simplified Hilbert space version of Theorem 4.2 in [11]; see also [7, pp. 17, 18].

THEOREM 4.4. *Let A be a subspace satisfying*

(i) $T_0 \subset A \subset T_1, \qquad \dim(A/T_0) = d.$

Then

(ii) $T_0^+ \subset A^* \subset T_1^+, \qquad \dim(A^*/T_0^+) = t - d,$

and there exist subspaces M_1, M_1^+ such that

(iii) $M_1 \subset T_1 \ominus T_0, \qquad M_1^+ \subset T_1^+ \ominus T_0^+,$

$\dim M_1 = d, \qquad \dim M_1^+ = t - d,$

$\langle M_1, M_1^+\rangle = 0,$

and

$$\text{(iv)}\quad A = T_0 \oplus M_1, \qquad A^* = T_0^+ \oplus M_1^+,$$

$$\text{(v)}\quad A = T_1 \cap (M_1^+)^*, \qquad A^* = T_1^+ \cap M_1^*.$$

Conversely, if M_1, M_1^+ *satisfy* (iii) *then* $A = T_0 \oplus M_1$ *is a subspace satisfying* (i), *and* (ii), (iv), (v) *are valid.*

Proof. If A satisfies (i) we know $M_1 = A \ominus T_0 \subset T_1 \ominus T_0$ is such that $\dim M_1 = d$, and then $A = T_0 \oplus M_1$, $A^* = T_0^* \cap M_1^* = T_1^+ \cap M_1^*$. Now $T_1^* = T_0^+ \subset A^* \subset T_0^* = T_1^+$, and $A^* = T_0^+ \oplus M_1^+$ for some subspace $M_1^+ \subset T_1^+ \ominus T_0^+$. We have $T_1 \ominus T_0 = M_1 \oplus M_2$, $T_1^+ \ominus T_0^+ = M_1^+ \oplus M_2^+$ for some subspaces M_2, M_2^+, and since $JM_1 = M_2^+$, $JM_2 = M_1^+$, we see that $\dim(A^*/T_0^+) = \dim M_1^+ = \dim M_2 = t - d$. The relation $\langle A, A^* \rangle = 0$ implies $\langle M_1, M_1^+ \rangle = 0$. Finally, $A^* = T_0^+ \oplus M_1^+$ implies $A = A^{**} = (T_0^+)^* \cap (M_1^+)^* = T_1 \cap (M_1^+)^*$. Conversely, let M_1, M_1^+ satisfy (iii) and let $A = T_0 \oplus M_1$. Then clearly A satisfies (i), and so (ii) is valid. From the relations $\langle M_1, M_1^+ \rangle = 0$, $M_1^+ \subset T_1^+ \ominus T_0^+$, $\langle T_0, T_0^+ \rangle = 0$, we have $T_0^+ \oplus M_1^+ \subset A^*$, and, since $\dim(A^*/T_0^+) = t - d = \dim M_1^+$, we have $A^* = T_0^+ \oplus M_1^+$. Then (v) follows.

The descriptions of A, A^* given by (v) above show how A, A^* are obtained from T_1, T_1^+ by the imposition of generalized boundary conditions. Thus

$$A = T_1 \cap (M_1^+)^* = \{w \in T_1 \mid \langle w, m_1^+ \rangle = 0\},$$

where $m_1^+ = (m_{11}^+, \ldots, m_{1t-d}^+)$ is an $m \times (t - d)$ matrix whose elements form a basis for M_1^+.

It is important to note that in general A, A^* will contain nonsmooth elements. To see this, assume that $\Re(T_1 - lI) = \mathfrak{H}$ for some $l \in \mathbb{C}$, i.e., $\nu(T_0^+ - \bar{l}I) = \{0\}$. (This will always be true for $l \in \mathbb{C}_0$ in the symmetric case $T_0 = T_0^+$.) Then given any $g \in \mathfrak{H}$ there exists an $f \in \mathfrak{D}(T_1)$ such that $\{f, g\} \in T_1 - lI$, or $\{f, g + lf\} \in T_1$, or, using (4.6),

$$(f, (L^+ - \bar{l}M)\varphi)_2 = (g, M\varphi)_2, \qquad \varphi \in C_0^\infty(\iota).$$

Now if $f \in AC_{\text{loc}}^{\nu-1}(\iota)$ then

$$((L - lM)f, \varphi)_2 = (g, M\varphi)_2, \qquad \varphi \in C_0^\infty(\iota),$$

and then the Corollary of Theorem 2.2 implies that $g \in AC_{\text{loc}}^{\nu-1}(\iota)$ and $Mg = (L - lM)f$. Therefore, if $\mathfrak{H}$ contains $g \notin AC_{\text{loc}}^{\nu-1}(\iota)$ there exist $f \in \mathfrak{D}(T_1)$ which do not satisfy $f \in AC_{\text{loc}}^{\nu-1}(\iota)$. In case $\iota = (a, b)$ is a finite interval, and L, M are regular on ι, it can be shown that $\mathfrak{H} = \mathfrak{H}_H$ consists of $f \in AC^{\mu-1}(\iota)$, where $f^{(\mu)} \in L^2(\iota)$ and f satisfies a set of homogeneous boundary conditions involving only f,

$f', \ldots, f^{(\mu-1)}$ at a and b. (This situation will be treated in detail in a subsequent paper.) Thus, in this regular case, if $\mu \geqslant 1$ and $T_0 = T_0^+$, then $\mathfrak{D}(T_1)$ will contain $f \notin AC_{\mathrm{loc}}^{r-1}(\iota)$. This implies $\mathfrak{D}(T_0)$ contains such f, for if $\mathfrak{D}(T_0) \subset AC_{\mathrm{loc}}^{r-1}(\iota)$, since $\mathfrak{D}(T_1 \ominus T_0) \subset C^r(\bar{\iota})$, we would have $\mathfrak{D}(T_1) \subset AC_{\mathrm{loc}}^{r-1}(\iota)$. Therefore, *every* subspace extension A of T_0 in $\mathfrak{H}^2$ must contain in its domain $f \notin AC_{\mathrm{loc}}^{r-1}(\iota)$. In [3] Brauer, in discussing boundary conditions in the case $T_0 = T_0^+$, $n > \nu$, $\mathfrak{H} = \mathfrak{H}_H$, where $H = M_F$, the Friedrichs extension of M_0, assumed the existence of a self-adjoint restriction A of T_1, whose domain $\mathfrak{D}(A) \subset AC_{\mathrm{loc}}^{r-1}(\iota)$. As we have seen above, no such restriction A exists if $\mu \geqslant 1$ even in the regular case. This shows that in general A, A^* can not be described by boundary conditions of the usual type.

A concrete example of this problem is given by $L = D^4$, $M = -D^2$, $m = 1$, on $\iota = (0, 1)$, and $H = M_F$, where

$$\mathfrak{D}(M_F) = \{f \in \mathfrak{D}(M_{\max}) \mid f(0) = f(1) = 0\}.$$

Thus $r = n = 4 > \nu = 2$. The construction of $\mathfrak{H} = \mathfrak{H}_H$ was carried out in Section 3 (there $H = H_{\infty\infty}$ and $\mathfrak{H}_H = \mathfrak{H}_{\infty\infty}$), and it turned out that

$$\mathfrak{H} = \{f \in AC([0, 1]) \mid f' \in L^2(\iota), f(0) = f(1) = 0\}.$$

Let h be any function satisfying $h \in C([0, 1])$ but $h \notin AC_{\mathrm{loc}}(\iota)$, and put

$$g(x) = \int_0^x h(t)\, dt - \left(\int_0^1 h(t)\, dt\right) x.$$

Then $g \in \mathfrak{H}$ but $g \notin AC_{\mathrm{loc}}^1(\iota)$.

In many situations (for example, the regular case) it is possible to show that smooth versions of subspace extensions A of T_0 satisfy boundary conditions of the usual type. There always exist smooth versions of subspace extensions A satisfying $T_0 \subset A \subset T_1$.

THEOREM 4.5. *Let A, A^* be an adjoint pair of subspaces satisfying*

$$T_0 \subset A \subset T_1, \qquad T_0^+ \subset A^* \subset T_1^+, \qquad \dim(A/T_0) = \dim(T_1^+/A^*) = d.$$

If $A_T = A \cap T$, $A_T^+ = A^* \cap T^+$, then A_T, $A_T^+ \subset C^r(\iota) \times C^\nu(\iota)$, and $(A_T)^c = A$, $(A_T^+)^c = A^*$.

Proof. Since $A_T \subset T$, $A_T^+ \subset T^+$ we have A_T, $A_T^+ \subset C^r(\iota) \times C^\nu(\iota)$. Now $S \oplus (A \ominus T_0) \subset S \oplus (T_1 \ominus T_0) \subset T$ and $S \oplus (A \ominus T_0) \subset A$ imply that $S \oplus (A \ominus T_0) \subset A \cap T = A_T \subset A$, and so $A = S^c \oplus (A \ominus T_0) \subset (S \oplus (A \ominus T_0))^c \subset (A_T)^c \subset A$, or $(A_T)^c = A$. Similarly $(A_T^+)^c = A^*$.

5. Generalized Resolvents of Minimal Subspaces

Let $\mathfrak{K}$ be any Hilbert space such that $\mathfrak{H} = \mathfrak{H}_H \subset \mathfrak{K}$ ($\mathfrak{H} = \mathfrak{K}$ is not excluded), and suppose A is a subspace in $\mathfrak{K}^2$ such that

$$T_0 \subset A \subset \mathfrak{K}^2, \qquad T_0^+ \subset A^* \subset \mathfrak{K}^2.$$

Here we are still assuming H is such that (3.10), (3.11) are valid with $S = H$. The resolvent set $\rho(A)$ of A is defined as the set of all $l \in \mathbb{C}$ such that $(A - lI)^{-1}$ exists as a (bounded) operator on all of $\mathfrak{K}$. It is an open set; suppose it is not empty. We define the *generalized resolvent* R_A *of* T_0 *corresponding to the extension* A via

$$R_A(l)h = P(A - lI)^{-1}h, \qquad h \in \mathfrak{H},\ l \in \rho(A),$$

where P is the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. In case $\mathfrak{K} = \mathfrak{H}$, R_A is the resolvent of A. It is the object of this section to show that $R_A(l)G$ is an integral operator with a smooth kernel.

The equality $((A - lI)^{-1})^* = (A^* - \bar{l}I)^{-1}$ readily implies that $(R_A(l))^* = R_{A^*}(\bar{l})$. If

$$T_A(l) = (R_A(l))^{-1} + lI = \{\{R_A(l)h, h + lR_A(l)h\} \mid h \in \mathfrak{H}\},$$

then $T_0 \subset T_A(l)$, since if $\{f, g\} \in T_0 \subset A$ we have $\{g - lf, f\} \in (A - lI)^{-1}$, and if $h = g - lf \in \mathfrak{H}$ then $f = R_A(l)h$, $g = h + lR_A(l)h$. Similarly, $T_0^+ \subset A^*$ implies that

$$T_0^+ \subset T_{A^*}(\bar{l}) = (R_{A^*}(\bar{l}))^{-1} + \bar{l}I = (T_A(l))^*,$$

and hence $T_A(l) \subset (T_0^+)^* = T_1$, so that we have

$$T_0 \subset T_A(l) \subset T_1, \qquad T_0^+ \subset T_{A^*}(\bar{l}) \subset T_1^+.$$

Since $T_A(l) \subset T_1 = (T_0^+)^*$ we have that for all $h \in \mathfrak{H}$ and $\varphi \in C_0^\infty(\iota)$,

$$\begin{aligned} 0 &= (h + lR_A(l)h, \varphi) - (R_A(l)h, GL^+\varphi) \\ &= (h + lR_A(l)h, M\varphi)_2 - (R_A(l)h, L^+\varphi)_2, \end{aligned}$$

and this is true for $h = G\alpha$, $\alpha \in L_0^2(\iota)$, so that

$$\begin{aligned} (R_A(l)G\alpha, (L^+ - \bar{l}M)\varphi)_2 &= (G\alpha, M\varphi)_2 = (MG\alpha, \varphi)_2 \\ &= (\alpha, \varphi)_2, \qquad \alpha \in L_0^2(\iota),\ \varphi \in C_0^\infty(\iota). \end{aligned}$$

From the Corollary to Theorem 2.2 it now follows that $R_A(l)G\alpha \in AC_{\mathrm{loc}}^{r-1}(\iota)$ and

$$(L - lM)\, R_A(l)G\alpha = MG\alpha = \alpha \tag{5.1}$$

for all those $l \in \rho(A)$ such that the leading coefficient of $L^+ - \bar{l}M$ is invertible on all of ι. This is the same as the set of $l \in \rho(A)$ where the leading coefficient of $L - lM$ is invertible on all of ι, i.e., $l \in \rho(A) \cap (\mathbb{C}\backslash\mathbb{C}_e)$, where $\mathbb{C}_e$ is the exceptional set appearing in Theorem 4.3(ii). We note that (5.1) shows that if $\alpha \in C(\iota) \cap L_0^2(\iota)$, then $\{R_A(l)G\alpha, G\alpha + lR_A(l)G\alpha\} \in T_A(l) \cap T$, for $R_A(l)G\alpha \in C^r(\iota)$.

The variation of constants formula yields another solution of the equation $(L - lM)u = \alpha$, namely

$$(L - lM)\, R_0(l)G\alpha = MG\alpha = \alpha, \tag{5.2}$$

where $R_0(l)G$ is an integral operator

$$R_0(l)\, G\alpha(x) = \int_\iota k_0(x, y, l)\, \alpha(y)\, dy, \qquad x \in \iota.$$

Specifically, we define

$$\begin{aligned} k_0(x, y, l) &= \tfrac{1}{2}s(x, l)(\mathscr{S}(l))^{-1}(s^+(y, \bar{l}))^*, \qquad & y \leqslant x, \\ &= -\tfrac{1}{2}s(x, l)(\mathscr{S}(l))^{-1}(s^+(y, l))^*, \qquad & y > x. \end{aligned}$$

Here $s(l) = (s_1(l),\ldots, s_{rm}(l))$, $s^+(\bar{l}) = (s_1^+(\bar{l}),\ldots, s_{rm}^+(\bar{l}))$ satisfy

$$(L - lM)\, s(l) = 0, \qquad (L^+ - \bar{l}M)\, s^+(\bar{l}) = 0, \qquad \hat{s}(c, l) = \hat{s}^+(c, \bar{l}) = I_{rm}\,,$$

for some $c \in \iota$. The matrix-valued functions $s(l)$, $s^+(\bar{l})$ constitute bases for the solutions of the homogeneous equations $(L - lM)u = 0$, $(L^+ - \bar{l}M)u^+ = 0$, which are analytic in l and $\bar{l}$, respectively, for $l \in \mathbb{C}\backslash\mathbb{C}_e$. The matrix $\mathscr{S}(l)$ is given by

$$\mathscr{S}(l) = [s(l), s^+(\bar{l})](x) = (\hat{s}^+(x, \bar{l}))^*\, \mathscr{B}_{L-lM}(x)\, \hat{s}(x, l).$$

It follows from Green's formula that $\mathscr{S}(l)$ is independent of $x \in \iota$, $\det \mathscr{S}(l) = \det \mathscr{B}_{L-lM}(c) = (\det P_r(c, l))^r$, where $P_r(x, l)$ is the leading coefficient of $L - lM$. Thus $\mathscr{S}(l)$ is invertible for all $l \in \mathbb{C}\backslash\mathbb{C}_e$.

Analogous to (5.1) and (5.2) we have

$$(L^+ - \bar{l}M)\, R_{A^*}(\bar{l})G\beta = MG\beta = \beta, \qquad \beta \in L_0^2(\iota), \tag{5.3}$$

$$(L^+ - \bar{l}M)\, R_0^+(\bar{l})G\beta = MG\beta = \beta, \qquad \beta \in L_0^2(\iota), \tag{5.4}$$

where

$$R_0^+(\bar{l})\, G\beta(x) = \int_\iota k_0^+(x, y, \bar{l})\, \beta(y)\, dy, \qquad x \in \iota,$$

and

$$\begin{aligned} k_0^+(x, y, \bar{l}) &= \tfrac{1}{2}s^+(x, \bar{l})(\mathscr{S}^+(\bar{l}))^{-1}(s(y, l))^*, \qquad & x \geqslant y, \\ &= -\tfrac{1}{2}s^+(x, \bar{l})(\mathscr{S}^+(\bar{l}))^{-1}(s(y, l))^*, \qquad & x < y, \end{aligned}$$

with

$$\mathscr{S}^+(\bar{l}) = [s^+(\bar{l}), s(l)](x) = \tilde{s}^*(x, l)\, \mathscr{B}_{L^+-\bar{l}M}(x)\, \tilde{s}^+(x, \bar{l})$$

which is independent of $x \in \iota$. It also follows from the Green's formula that

$$(\mathscr{B}_{L-lM}(x))^* = -\mathscr{B}_{L^+-\bar{l}M}(x),$$

and hence

$$(\mathscr{S}(l))^* = -\mathscr{S}^+(\bar{l}),$$

which implies that

$$k_0^*(x, y, l) = k_0^+(y, x, \bar{l}),$$

and also

$$(R_0(l)G\alpha, \beta)_2 = (\alpha, R_0^+(\bar{l})G\beta)_2, \qquad \alpha, \beta \in L_0^2(\iota). \tag{5.5}$$

From (5.1) and (5.2) we see that $R_A(l)G\alpha - R_0(l)G\alpha$ satisfies the homogeneous equation $(L - lM)u = 0$, and thus there is a constant $rm \times 1$ matrix $c(\alpha, l)$ such that

$$R_A(l)G\alpha = R_0(l)G\alpha + s(l)\, c(\alpha, l), \qquad \alpha \in L_0^2(\iota),\ l \in \rho(A) \cap (\mathbb{C}\backslash\mathbb{C}_e).$$

Now we claim that $c(\alpha, l) = (\alpha, a(l))_2$ for some $a(l) \in L^2_{\text{loc}}(\iota)$ which is independent of α. To see this let h denote a real-valued function on ι, with $h \in C_0^\infty(J)$ for some closed bounded interval J containing c in its interior, and

$$h(c) = 1, \qquad 0 \leqslant h(c) \leqslant 1, \qquad x \in \iota.$$

Then we define the $m \times rm$ matrix-valued function $\tilde{h}^+ \in C_0^\infty(\iota)$ by

$$\tilde{h}^+ = (hI_m : (-1)\, h'I_m : \ldots : (-1)^{r-1}\, h^{(r-1)}I_m).$$

This has the property that

$$(f, \tilde{h}^+)_2 = (\tilde{f}, hI_{rm})_2, \qquad f \in C^{r-1}(\iota).$$

Thus

$$\begin{aligned}(\alpha, (R_A(l))^*G\tilde{h}^+)_2 &= (G\alpha, (R_A(l))^*G\tilde{h}^+)\\ &= (R_A(l)G\alpha, G\tilde{h}^+) = (R_A(l)G\alpha, \tilde{h}^+)_2\\ &= (R_0(l)G\alpha, \tilde{h}^+)_2 + (s(l), \tilde{h}^+)_2\, c(\alpha, l)\\ &= (\alpha, R_0^+(\bar{l})G\tilde{h}^+)_2 + (\tilde{s}(l), hI_{rm})_2\, c(\alpha, l).\end{aligned}$$

Since $\tilde{s}(c, l) = I_{rm}$, if the length $|J|$ of J is small enough, $(\tilde{s}(l), hI_{rm})_2$ is invertible. Choose $|J|$ so that this is true. Then we see that

$$\begin{aligned} c(\alpha, l) &= ((\tilde{s}(l), hI_{rm})_2)^{-1}[(\alpha, R_{A^*}(\bar{l})G\tilde{h}^+)_2 - (\alpha, R_0{}^+(\bar{l})G\tilde{h}^+)_2] \\ &= (\alpha, a(l))_2 , \end{aligned}$$

where

$$a(l) = [R_{A^*}(\bar{l})G\tilde{h}^+ - R_0{}^+(\bar{l})G\tilde{h}^+]((hI_{rm}, \tilde{s}(l))_2)^{-1}.$$

Clearly $a(l) \in L^2_{\mathrm{loc}}(\iota)$, and, in fact, due to (5.3) and (5.4) we see that $(L^+ - \bar{l}M)\, a(l) = 0$, so that

$$a(l) = s^+(\bar{l})\, \Psi^*(l),$$

for some $rm \times rm$ matrix $\Psi(l)$. Therefore

$$R_A(l)G\alpha = R_0(l)G\alpha + s(l)\, \Psi(l)(\alpha, s^+(\bar{l}))_2 , \qquad \alpha \in L_0{}^2(\iota).$$

Interchanging the roles of L and L^+, and l and $\bar{l}$, we obtain the representation for $R_{A^*}(\bar{l})$:

$$R_{A^*}(\bar{l})G\beta = R_0{}^+(\bar{l})G\beta + s^+(\bar{l})\, \Psi^+(\bar{l})(\beta, s(l))_2 , \qquad \beta \in L_0{}^2(\iota).$$

THEOREM 5.1. *Let $\mathfrak{K}$ be a Hilbert space containing $\mathfrak{H} = \mathfrak{H}_H$, and let A be a subspace in $\mathfrak{K}^2$ satisfying*

$$T_0 \subset A \subset \mathfrak{K}^2, \qquad T_0{}^+ \subset A^* \subset \mathfrak{K}^2,$$

and whose resolvent set $\rho(A)$ is not empty. For $l \in \rho(A) \cap (\mathbb{C}\backslash\mathbb{C}_e)$ the generalized resolvent $R_A(l)$ of T_0 corresponding to A is such that $R_A(l)G : L_0{}^2(\iota) \to \mathfrak{H}$ is an integral operator,

$$R_A(l)\, G\alpha(x) = \int_\iota k(x, y, l)\, MG\alpha(y)\, dy = \int_\iota k(x, y, l)\, \alpha(y)\, dy, \qquad \alpha \in L_0{}^2(\iota),$$

with kernel k given by

$$k(x, y, l) = k_0(x, y, l) + k_1(x, y, l),$$

where

$$k_1(x, y, l) = s(x, l)\, \Psi(l)(s^+(y, \bar{l}))^*.$$

Similarly, for $(R_A(l))^* = R_A{}^*(\bar{l})$ we have

$$R_{A^*}(\bar{l})\, G\beta(x) = \int_\iota k^+(x, y, \bar{l})\, \beta(y)\, dy, \qquad \beta \in L_0{}^2(\iota),$$

with

$$k^+(x, y, \bar{l}) = k_0^+(x, y, \bar{l}) + k_1^+(x, y, \bar{l})$$

and

$$k_1^+(x, y, \bar{l}) = s^+(x, \bar{l})\, \Psi^+(\bar{l})(s(y, l))^*.$$

The matrix Ψ *satisfies*

$$\Psi^*(l) = \Psi^+(\bar{l}),$$

and Ψ *is analytic for* $l \in \rho(A) \cap (\mathbb{C} \backslash \mathbb{C}_e)$, *so that*

$$k^+(x, y, \bar{l}) = k^*(y, x, l).$$

Proof. We have proved everything except the statements concerning the nature of Ψ. For $\alpha, \beta \in L_0^2(\iota)$ we have

$$\begin{aligned}(R_A(l)\, G\alpha, G\beta) &= (R_A(l)\, G\alpha, \beta)_2 \\ &= (R_0(l)\, G\alpha, \beta)_2 + (s(l), \beta)_2 \Psi(l)(\alpha, s^+(\bar{l}))_2 ,\end{aligned} \tag{5.6}$$

and putting $\alpha = \beta = \tilde{h}^+$ we obtain

$$(\tilde{s}(l), hI_{rm})_2\, \Psi(l)(hI_{rm}, \tilde{s}^+(\bar{l}))_2 = (R_A(l)G\tilde{h}^+, \tilde{h}^+)_2 - (R_0(l)G\tilde{h}^+, \tilde{h}^+)_2 .$$

Thus, if $|J|$ is so small that both $(\tilde{s}(l), hI_{rm})_2$ and $(hI_{rm}, \tilde{s}^+(l))_2$ are invertible, we have

$$\Psi(l) = ((\tilde{s}(l), hI_{rm})_2)^{-1}[(R_A(l)G\tilde{h}^+, \tilde{h}^+)_2 - (R_0(l)G\tilde{h}^+, \tilde{h}^+)_2]((hI_{rm}, \tilde{s}^+(\bar{l}))_2)^{-1}. \tag{5.7}$$

Now $(\tilde{s}(l), hI_{rm})_2$ and $(hI_{rm}, \tilde{s}^+(\bar{l}))_2$ are analytic in $l \in \mathbb{C} \backslash \mathbb{C}_e$, and $(R_A(l)G\tilde{h}^+, \tilde{h}^+)_2 = (R_A(l)G\tilde{h}^+, G\tilde{h}^+)$ is analytic for $l \in \rho(A)$. From the explicit construction of $R_0(l)$ it follows that $(R_0(l)G\tilde{h}^+, \tilde{h}^+)_2$ is analytic for $l \in \mathbb{C} \backslash \mathbb{C}_e$. Therefore (5.7) shows that Ψ is analytic on the open set $\rho(A) \cap (\mathbb{C} \backslash \mathbb{C}_e)$.

In addition to (5.6) we have the alternate expression

$$\begin{aligned}(R_A(l)G\alpha, G\beta) &= (G\alpha, R_{A^*}(\bar{l})G\beta) = (\alpha, R_{A^*}(\bar{l})G\beta)_2 \\ &= (\alpha, R_0^+(\bar{l})G\beta)_2 + (s(l), \beta)_2(\Psi^+(\bar{l}))^*(\alpha, s^+(\bar{l}))_2 ,\end{aligned}$$

and, using (5.5) and (5.6), we get

$$(s(l), \beta)_2\, \Psi(l)(\alpha, s^+(\bar{l}))_2 = (s(l), \beta)_2(\Psi^+(\bar{l}))^*(\alpha, s^+(\bar{l}))_2$$

for all $\alpha, \beta \in L_0^2(\iota)$. Letting $\alpha = \beta = \tilde{h}^+$ we obtain $\Psi^*(l) = \Psi^+(\bar{l})$, completing the proof of the theorem.

Remark. The result of Theorem 5.1 can be used to show that $R_A(l)$ is an integral operator of Carleman type in case $M = I$ (multiplication by I_m). To see this we define for $x \in \iota$ the matrix-valued functions δ_x, η_x on ι so that $\delta_x \in C^n(J)$, where $J = [x, y] \subset \iota$ for some $y > x$, and

$$\begin{aligned}
\tilde{\delta}_x(t) &= 0_{mn}^{mn}, & a < t < x,\\
&= -((\mathscr{B}_L(x))^{-1})^*, & t = x,\\
&= 0_{mn}^{mn}, & y \leqslant t < b,\\
\eta_x(t) &= 0_{mn}^{mn}, & a < t < x,\\
&= L^+\delta_x(t), & x \leqslant t \leqslant y,\\
&= 0_{mn}^{mn}, & y < t < b.
\end{aligned}$$

Then $\delta_x, \eta_x \in L^2(\iota)$, and for all $f \in \mathfrak{D}(L_{\max})$,

$$\begin{aligned}
\langle \{f, Lf\}, \{\delta_x, \eta_x\}\rangle &= (Lf, \delta_x)_2 - (f, \eta_x)_2\\
&= (Lf, \delta_x)_{2,J} - (f, L^+\delta_x)_{2,J}\\
&= (\tilde{\delta}_x(y))^* \mathscr{B}_L(y) \tilde{f}(y) - (\tilde{\delta}_x(x))^* \mathscr{B}_L(x) \tilde{f}(x)\\
&= \tilde{f}(x).
\end{aligned}$$

Now $\{R_A(l)h, h + lR_A(l)h\} \in T_1 = L_{\max}$ for $h \in \mathfrak{H} = L^2(\iota)$, $l \in \rho(A)$, and hence

$$\begin{aligned}
\widetilde{R_A(l)\, h}(x) &= \langle \{R_A(l)h, h + lR_A(l)h\}, \{\delta_x, \eta_x\}\rangle\\
&= (lR_A(l)h + h, \delta_x)_2 - (R_A(l)h, \eta_x)_2\\
&= (h, (\bar{l}R_{A^*}(\bar{l}) + I)\,\delta_x - R_{A^*}(\bar{l})\,\eta_x)_2 .
\end{aligned}$$

Specializing this to $h = \alpha \in L_0^2(\iota)$ and comparing this result with that for $R_A(l)$ (note that $G = I$ in the case $M = I$), we find that

$$\widetilde{R_A(l)\,\alpha}(x) = \int_\iota \tilde{k}_x(x, y, l)\,\alpha(y)\,dy, \qquad \alpha \in L_0^2(\iota),$$

where

$$\tilde{k}_x = \begin{pmatrix} k \\ \partial k/\partial x \\ \vdots \\ \partial^{n-1}k/\partial x^{n-1} \end{pmatrix}$$

is such that

$$(\tilde{k}_x)^* = (\bar{l}R_{A^*}(\bar{l}) + I)\delta_x - R_{A^*}(\bar{l})\eta_x \in L^2(\iota),$$

i.e.,

$$\int_\iota \left| \left(\frac{\partial}{\partial x}\right)^j k(x, y, l) \right|^2 dy < \infty, \qquad x \in \iota,\ j = 0,\dots, n-1.$$

Since $k^+(x, y, \bar{l}) = k^*(y, x, l)$, a similar result is valid for the kernel k^+ of $R_{A^*}(\bar{l})$.

A simplified version of the technique used to prove Theorem 5.1 can be employed to show that the injection $G: L_0^2(\iota) \to \mathfrak{H} = \mathfrak{H}_H$ is an integral operator with a nice kernel. Theorem 3.2(iii) shows that

$$MG\alpha = \alpha, \qquad \alpha \in L_0^2(\iota).$$

Another solution of the nonhomogeneous equation $Mu = \alpha$ is given by $u = R_M\alpha$, where R_M is the integral operator

$$R_M\alpha(x) = \int_\iota k_M(x, y)\, \alpha(y)\, dy, \qquad \alpha \in L_0^2(\iota), \qquad x \in \iota.$$

Here the kernel k_M bears the same relationship to M as the kernel k_0 does to $L - lM$, so that if $s_M = (s_{M1}, \ldots, s_{M\nu m})$ is a basis for the solutions of $Mu = 0$ on ι,

$$\begin{aligned} k_M(x, y) &= \tfrac{1}{2} s_M(x)\, \mathscr{S}_M^{-1} s_M^*(y), \qquad & y \leqslant x, \\ &= -\tfrac{1}{2} s_M(x)\, \mathscr{S}_M^{-1} s_M^*(y), \qquad & y > x. \end{aligned}$$

The matrix $\mathscr{S}_M$ is given by

$$\mathscr{S}_M = \tilde{s}_M^*(x)\, \mathscr{B}_M(x)\, \tilde{s}_M(x),$$

which is independent of x. Here $\tilde{s}_M$ is the $\nu m \times \nu m$ matrix

$$\tilde{s}_M = \begin{pmatrix} s_M \\ s_M' \\ \vdots \\ s_M^{(\nu-1)} \end{pmatrix},$$

and $\mathscr{B}_M$ is defined for M as $\mathscr{B}_L$ was for L. Now the symmetry $M = M^+$ implies that

$$\mathscr{B}_M^* = -\mathscr{B}_M, \qquad \mathscr{S}_M^* = -\mathscr{S}_M, \qquad k_M^*(x, y) = k_M(y, x),$$

and

$$(R_M\alpha, \beta)_2 = (\alpha, R_M\beta)_2, \qquad \alpha, \beta \in L_0^2(\iota). \tag{5.8}$$

Analogous to (5.8) we have

$$(G\alpha, \beta)_2 = (G\alpha, G\beta) = (\alpha, G\beta)_2, \qquad \alpha, \beta \in L_0^2(\iota). \tag{5.9}$$

Now $M(G\alpha - R_M\alpha) = 0$ implies that for some constant $\nu m \times 1$ matrix $c(\alpha)$ we have

$$G\alpha = R_M\alpha + s_M c(\alpha), \qquad \alpha \in L_0^2(\iota). \tag{5.10}$$

Then we have from (5.8) and (5.9)

$$(G\alpha - R_M\alpha, \beta)_2 = (s_M, \beta)_2\, c(\alpha) = (\alpha, G\beta - R_M\beta)_2, \qquad \alpha, \beta \in L_0^2(\iota).$$

If we put $\beta = \chi_J s_M$, where χ_J is the characteristic function of a compact interval $J \subset \iota$, we see that $(s_M, \beta)_2 = (s_M, s_M)_{2,J}$ is an invertible Hermitian $\nu m \times \nu m$ matrix since it is just the Gramian matrix on J of the basis s_M. Therefore

$$\begin{aligned} c(\alpha) &= ((s_M, s_M)_{2,J})^{-1}(\alpha, G\chi_J s_M - R_M\chi_J s_M)_2, \\ &= (\alpha, a)_2, \end{aligned} \tag{5.11}$$

where

$$a = (G\chi_J s_M - R_M\chi_J s_M)((s_M, s_M)_{2,J})^{-1}.$$

Since $M(G\chi_J s_M - R_M\chi_J s_M) = 0$ we see that $a = s_M\Gamma^*$ for some $\nu m \times \nu m$ matrix Γ. Hence (5.10) and (5.11) imply that

$$G\alpha = R_M\alpha + s_M\Gamma(\alpha, s_M)_2, \qquad \alpha \in L_0^2(\iota).$$

Now the symmetry relations (5.8), (5.9) show that $\Gamma = \Gamma^*$, and we have proved the following result.

THEOREM 5.2. *The injection $G: L_0^2(\iota) \to \mathfrak{H} = \mathfrak{H}_H$ is an integral operator*

$$G\alpha(x) = \int_\iota g(x, y)\,\alpha(y)\,dy, \qquad \alpha \in L_0^2(\iota), \qquad x \in \iota,$$

with the kernel

$$g(x, y) = k_M(x, y) + g_1(x, y),$$

where

$$g_1(x, y) = s_M(x)\,\Gamma s_M^*(y),$$

and $\Gamma = \Gamma^*$.

6. The Symmetric Case

The case when the minimal operator S is symmetric in $\mathfrak{H} = \mathfrak{H}_H$ is of special interest.

THEOREM 6.1. *The minimal operator S is symmetric $S \subset S^*$, if and only if $L = L^+$.*

Proof. If $L = L^+$ then $S = S^+$, $T = T^+$, and hence $S \subset T \subset T^c = S^*$, from (4.7) and Theorem 4.2(iic). Conversely, suppose $S \subset S^*$.

Then for all $\{\varphi, \psi\}, \{\zeta, \eta\} \in S$ we have

$$\begin{aligned} 0 &= (\psi, \zeta) - (\varphi, \eta) = (\psi, M\zeta)_2 - (M\varphi, \eta)_2 \\ &= (M\psi, \zeta)_2 - (\varphi, M\eta)_2 = (L\varphi, \zeta)_2 - (\varphi, L\zeta)_2 \\ &= (L\varphi, \zeta)_2 - (L^+\varphi, \zeta)_2 . \end{aligned}$$

Therefore $L\varphi = L^+\varphi$ for all $\varphi \in C_0^\infty(\iota)$, and $L = L^+$.

COROLLARY. *S is symmetric if and only if*

$$P_k = \sum_{j=k}^{n} (-1)^j \binom{j}{k} D^{j-k} P_j^*, \qquad k = 0,\ldots, n.$$

We now *assume* $L = L^+$, and summarize the results of Sections 4 and 5 in this symmetric case. We have

$$\begin{aligned} T &= \{\{f, g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), g \in C^\nu(\iota), Lf = Mg\}, \\ S &= \{\{\varphi, \psi\} \in \mathfrak{H}^2 \mid \varphi \in C_0^\infty(\iota), \psi = GL\varphi\} \\ &= \{\{\varphi, \psi\} \in T \mid \varphi \in C_0^\infty(\iota), (\psi, f) = (M\psi, f)_2 \text{ for all } f \in \mathfrak{N}_H \\ &\qquad \text{(and for all } f \in \mathfrak{H})\}, \end{aligned}$$

$$S^* = \{\{f, g\} \in \mathfrak{H}^2 \mid (g, M\varphi)_2 = (f, L\varphi)_2 , \text{ all } \varphi \in C_0^\infty(\iota)\},$$

$$S \subset T, \qquad T_0 \subset S^c \subset T^c = T_1 = T_0^* = S^*,$$

$$T_1 \ominus T_0 = T \cap JT = \{\{f, g\} \in \mathfrak{H}^2 \mid f, g \in C^r(\iota), Lf = Mg, Mf = -Lg\},$$

$$\mathfrak{D}(T_1 \ominus T_0) = \mathfrak{R}(T_1 \ominus T_0) = \nu(T^2 + I),$$

$$T_1(0) = T(0) = \mathfrak{N}_H = \{f \in C^\nu(\iota) \cap \mathfrak{H} \mid Mf = 0\},$$

and

$$\nu(T_1 - lI) = \nu(T - lI) = \{f \in C^r(\iota) \cap \mathfrak{H} \mid (L - lM)f = 0\}, \quad l \in \mathbb{C}\backslash\mathbb{C}_e , \quad (6.1)$$

whereas

$$T - lI = \{\{f, g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), g \in C^\nu(\iota), (L - lM)f = Mg\}, \qquad l \in \mathbb{C}.$$

For the case $L = L^+$ the exceptional set $\mathbb{C}_e \subset \mathbb{R}$. This is clear when $n \neq \nu$. Since $L = L^+$ we must have $P_n^* = (-1)^n P_n$ and if $n = \nu = 2\mu$, $P_\nu^* = P_\nu$. This implies that

$$\sigma(Q_\nu^{-1}(x)\, P_\nu(x)) = \{l \in \mathbb{C} \mid \det(P_\nu(x) - lQ_\nu(x)) = 0\} \subset \mathbb{R}.$$

For suppose $P_\nu(x)\xi = lQ_\nu(x)\xi$, for some $\xi \in \mathbb{C}^m$, $\xi \neq 0$. Then $\xi^*P_\nu(x)\xi = l\xi^*Q_\nu(x)\xi$, and, since $(-1)^\mu Q_\nu(x) > 0$, we have $\xi^*Q_\nu(x)\xi \neq 0$, or $l =$

$\xi^* P_\nu(x)\xi / \xi^* Q_N(x)\xi$, a real number. In particular, we note that (6.1) is valid for all $l \in \mathbb{C}_0 = \mathbb{C}\backslash\mathbb{R}$.

We know from general results about symmetric subspaces (closed linear manifolds) in $\mathfrak{H}^2$ [13, p. 92] that

$$T_1 = S^* = T_0 \dotplus M_S(l) \dotplus M_S(\bar{l}), \qquad l \in \mathbb{C}_0\text{, a direct sum,}$$

and there exist self-adjoint subspace extensions $A = A^*$ of S in $\mathfrak{H}^2$ if and only if for some (and hence for all) $l \in \mathbb{C}_0$,

$$\dim M_S(l) = \dim M_S(\bar{l}).$$

Here

$$M_S(l) = \{\{f, g\} \in S^* \mid g = lf\}$$

is clearly an operator, and since $\mathfrak{D}(M_S(l)) = \nu(S^* - lI) = \nu(T - lI)$, $l \in \mathbb{C}_0$, we see that S has self-adjoint extensions in $\mathfrak{H}^2$ if and only if

$$\dim \nu(T - lI) = \dim \nu(T - \bar{l}I), \qquad \text{some } l \in \mathbb{C}_0. \tag{6.2}$$

From (6.1) above we have

$$\dim \nu(T - lI) \leqslant rm, \qquad l \in \mathbb{C}_0.$$

Sufficient conditions for the validity of (6.2) are given in the following result.

THEOREM 6.2. *The minimal symmetric operator S has self-adjoint subspace extensions in $\mathfrak{H}^2$ if either*

(i) $(L\varphi, \varphi)_2 \geqslant c(M\varphi, \varphi)_2$ *for some* $c \in \mathbb{R}$, *all* $\varphi \in C_0^\infty(\iota)$, *or*

(ii) L, M *have real coefficients, that is,* $\bar{P}_k = P_k$, $k = 0,\ldots, n$, $\bar{Q}_k = Q_k$, $k = 0,\ldots, \nu$.

Proof. For $\{\varphi, \psi\} = \{\varphi, GL\varphi\} \in S$ we have

$$(S\varphi, \varphi) = (GL\varphi, \varphi) = (L\varphi, \varphi)_2 \geqslant c(M\varphi, \varphi)_2 = c(\varphi, \varphi), \quad \varphi \in \mathfrak{D}(S) = C_0^\infty(\iota).$$

Thus S is bounded below by c, and hence has at least one self-adjoint extension in $\mathfrak{H}^2$, namely, the Friedrichs extension [7, p. 39]. If (ii) is assumed, then $(L - lM)f = 0$ for some $f \in C^r(\iota) \cap \mathfrak{H}$ and $l \in \mathbb{C}_0$ implies that $(L - \bar{l}M)\bar{f} = 0$. Since $f \in \mathfrak{H}$ there exists a sequence $\varphi_n \in \mathfrak{D}(H)$ such that $\| f - \varphi_n \| \to 0$. Now $\| \varphi_n - \varphi_m \|^2 = (M(\varphi_n - \varphi_m), \varphi_n - \varphi_m)_2 \to 0$ implies that $\| \bar{\varphi}_n - \bar{\varphi}_m \| \to 0$, and there exists an element $g \in \mathfrak{H}$ such that $\| \bar{\varphi}_n - g \| \to 0$. It follows from (3.10) for $S = H$ that $\bar{f} = g \in \mathfrak{H}$. Hence, from (6.1) we see that $f \in \nu(T - lI)$ if and only if $\bar{f} \in \nu(T - \bar{l}I)$, and thus (6.2) is valid, showing that S has self-adjoint extensions in $\mathfrak{H}^2$.

The self-adjoint version of Theorem 4.4 is the following result, which is essentially Theorem 12 in [7].

THEOREM 6.3. *Let A be a self-adjoint subspace satisfying*

(i) $T_0 \subset A \subset T_1$, $\dim(A/T_0) = d$.

Then $2d = t = \dim(T_1/T_0)$, *and there exists a subspace* M_1 *such that*

(ii) $M_1 \subset T_1 \ominus T_0$, $\dim M_1 = d$, $\langle M_1, M_1 \rangle = 0$,

and

(iii) $A = T_0 \oplus M_1 = T_1 \cap (M_1)^*$.

Conversely, if M_1 *satisfies* (ii) *where* $2d = t$, *then* $A = T_0 \oplus M_1$ *is self-adjoint and satisfies* (i) *and* (iii).

Let $\mathfrak{K}$ be any Hilbert space satisfying $\mathfrak{K} \supset \mathfrak{H} = \mathfrak{H}_H$, and let A be a self-adjoint subspace in $\mathfrak{K}^2$ such that

$$S \subset T_0 \subset A = A^* \subset \mathfrak{K}^2.$$

Then $\mathbb{C}_0 \subset \rho(A)$, and the generalized resolvent R_A of T_0 corresponding to A is given by

$$R_A(l)h = P(A - lI)^{-1}h, \qquad h \in \mathfrak{H}, l \in \rho(A),$$

where P is the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. The equality $((A - lI)^{-1})^* = (A - \bar{l}I)^{-1}$ implies that $(R_A(l))^* = R_A(\bar{l})$, and we have $T_0 \subset T_A(l) \subset T_1 = T_0^*$. Moreover, Theorem 5.1 applied to this case when $T_0 = T_0{}^+$ shows that we may write

$$R_A(l)\, G\alpha(x) = \int_\iota k(x, y, l)\, \alpha(y)\, dy, \qquad \alpha \in L_0{}^2(\iota),$$

where the kernel k is given by

$$k(x, y, l) = k_0(x, y, l) + k_1(x, y, l),$$

with k_0 being the kernel of $R_0(l)$ and

$$k_1(x, y, l) = s(x, l)\, \Psi(l)(s(y, \bar{l}))^*.$$

We have

$$k_0^*(x, y, l) = k_0(y, x, \bar{l})$$

and

$$\Psi^*(l) = \Psi(\bar{l}), \qquad k_1^*(x, y, l) = k_1(y, x, \bar{l}).$$

The matrix Ψ is analytic on $\rho(A) \cap (\mathbb{C} \backslash \mathbb{C}_e) \supset \mathbb{C}_0$.

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